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M-theory and E_{10}: billiards, branes, and imaginary roots

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Abstract: Eleven dimensional supergravity compactified on T^{10} admits classical solutions describing what is known as *billiard cosmology* — a dynamics expressible as an abstract (billiard) ball moving in the 10-dimensional root space of the infinite dimensional Lie algebra E_{10}, occasionally bouncing off walls in that space. Unlike finite dimensional Lie algebras, E_{10} has negative and zero norm roots, in addition to the positive norm roots. The walls above are related to physical fluxes that, in turn, are related to positive norm roots (called *real roots*) of E_{10}. We propose that zero and negative norm roots, called *imaginary roots*, are related to physical branes. Adding "matter" to the billiard cosmology corresponds to adding potential terms associated to imaginary roots. The, as yet, mysterious relation between E_{10} and M-theory on T^{10} can now be expanded as follows: real roots correspond to fluxes or instantons, and imaginary roots correspond to particles and branes (in the cases we checked). Interactions between fluxes and branes and between branes and branes are classified according to the inner product of the corresponding roots (again in the cases we checked). We conclude with a discussion of an effective hamiltonian description that captures some features of M-theory on T^{10}.

Keywords: M-Theory, String Duality, Models of Quantum Gravity.

Contents

1. Introduction

Our setting is M-theory with all of space compactified by periodic boundary conditions. When more than $d = 8$ dimensions are compact, there is no notion of *moduli space of vacua*; the metric and even the topology of the compact directions should be allowed to fluctuate and should be treated quantum mechanically. But a complete quantum mechanical formulation of this setting is, of course, at the moment unknown.

It has been suggested over two decades ago that the infinite dimensional Kac-Moody Lie algebra E_{10} is relevant to the formulation of this theory [1]. Since then, the possible connection between M-theory and E_{10} has been discussed in various settings (see [2]–[24] for a sample). There are also recent conjectures about a formulation of *uncompactified* M-theory in terms of E_{10} [25] and about a description of the behavior of M-theory near spacelike singularities in terms of E_{10} [26]–[30]. E_{10} and other Kac-Moody and Generalized Kac-Moody algebras also appeared in other contexts in string theory (see for instance [31]–[35]) which we will not discuss here. Although a lot of progress has been made [36]–[40], a full understanding of the connection of M-theory to E_{10} is still an open problem.

One of the features that distinguish infinite dimensional Kac-Moody Lie algebras, such as E_{10}, from the finite dimensional ones is the existence of *imaginary roots* [41] in the root space. From the physical point of view these roots are mysterious, and to the best of our knowledge their physical interpretation has not been explored.

In this paper, we will study these imaginary roots from a physical perspective. We will propose that they can be matched with actual branes.

We find it convenient to work with periodic boundary conditions, although our proposal about the relation of imaginary roots and branes can be readily adapted to the noncompact setting of [26]–[30]. The simplest way to set periodic boundary conditions on all 10 spatial directions is to pick a topology of T^{10}. Classically, a homogeneous Kasner metric on T^{10} of the form

$$ds^2 = -dt^2 + \sum_{i=1}^{10} R_i(t)^2 dx_i^2\,, \qquad 0 \le x_i \le 2\pi\,, \qquad i = 1,\ldots,10, \tag{1.1}$$

can be a solution to Einstein's equations if all $\log R_i$'s are linear in $\log t$. We set the slope to be a constant p_i so that

$$\log\frac{R_i(t)}{R_i(t_0)} = p_i \log\frac{t}{t_0}\,, \qquad i = 1,\ldots,10\,. \tag{1.2}$$

Without matter, the Kasner metric (1.1) is a solution provided the constants $p_1,\ldots,p_{10}$ satisfy $\sum p_i = \sum p_i^2 = 1$. This metric describes a universe that is contracting in some directions (where $p_i < 0$) and expanding in other directions (where $p_i > 0$). This metric was extensively studied in [9], where it was shown that a classical treatment of a Kasner metric (1.1) is asymptotically trustworthy in the far future if the vector of powers $\vec{p} \equiv (p_1, \ldots, p_{10})$ describes a timelike vector in $\mathbb{R}^{9,1}$ (unrelated to the geometrical spacetime) with a suitably chosen metric

$$\|\vec{p}\|^2 \equiv \sum p_i^2 - \left(\sum p_i\right)^2 . \tag{1.3}$$

$\|\vec{p}\|^2 = 0$ if $\sum p_i = \sum p_i^2 = 1$, and thus $\vec{p}$ can never be timelike unless we also include matter. But before we add matter in the form of Kaluza-Klein particles and branes, let us discuss the dynamics in the presence of fluxes. A *flux* in this context could be either a constant $G = dC$ (where C is the 3-form of 11D supergravity) or a U-dual field. A U-dual field could describe, for example, a nontrivial fibration of one of the ten spatial directions over the remaining nine. The fluxes are quantized and have discrete values, and there are instanton effects that change the fluxes by integer amounts. Explicit constructions of such instanton terms appear in [42].

With fluxes, the classical dynamics of the scale factors $\log R_i$ is no longer linear in $\log t$. It was argued in [26]–[30] that the evolution of the logs of the scale factors can be approximated by a piecewise linear function that describes Kasner epochs separated by sharp changes in the vector $\vec{p}$. The changes correspond to reflections off (abstract) walls in $(\log R_i)$-space. The walls correspond to the various fluxes that are present. The orientation of each wall is determined by the type of flux, and its position is determined by the amount of flux. This evolution is called *billiard cosmology* since the dynamics is analogous to that of a billiard ball in an abstract 10-dimensional space with coordinates $\log R_i$, and the reflections are analogous to the ball bouncing off the walls. Even in the absence of fluxes the walls above are present quantum mechanically. They represent the necessary U-duality transformations that can be used to convert small dimensions to large dimensions [9]. Without matter, these reflections lead to a chaotic evolution [26].

E_{10} makes its appearance when we identify the ("billiard table") 10-dimensional space with the Cartan subalgebra of the infinite dimensional hyperbolic Kac-Moody Lie algebra, and identify each reflection off a wall with a fundamental reflection generator of the Weyl group. The metric (1.3) can be identified with the Cartan metric of E_{10} (which has signature $(9, 1)$).

The infinite dimensional noncompact group G_{10} that is defined as the exponential of a certain real form of the Lie algebra E_{10} is a natural extension of the finite dimensional noncompact groups $G_d = \exp E_d$ with $d \leq 8$ that appear as classical symmetry groups of the low energy limit of M-theory compactified on T^d. On the classical level, these symmetry groups are spontaneously broken, and G_d acts transitively on the moduli space of vacua whose metric and topology can be summarized by writing the moduli space as $\Gamma'_d \backslash G_d / K_d$. Here K_d is the maximal compact subgroup of G_d and $\Gamma'_d = \mathrm{SL}(d, \mathbb{Z}) \subset G_d$. For $d = 8$ we have $G_d \equiv E_{8(8)}(\mathbb{R})$, and $K_d = \mathrm{Spin}(16)/\mathbb{Z}_2$ [44]. On the quantum level, these groups are explicitly broken by loop and instanton effects, and are *not* good symmetries. This point is demonstrated in explicit formulas for low-energy effective scattering amplitudes (presented as terms in the low-energy effective action that contain, say, products of 4 curvature tensors) that appear in [42]. The quantum moduli space also contains extra

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identifications which extend Γ'_d to the full U-duality group Γ_d [5]. It is a discrete subgroup of G_d that preserves a lattice in an appropriate representation of G_d [45]. The extension of Γ'_d to Γ_d makes the volume of the moduli space finite. For $d = 8$ we have $\Gamma_d \equiv E_{8(8)}(\mathbb{Z})$. It is therefore also clear that $G_{10} \equiv \exp E_{10}$ cannot be an unbroken symmetry group of any formulation of M-theory on T^{10} that includes instanton effects. It has to be either explicitly or spontaneously broken.

Nevertheless, E_{10} provides a nice characterization of the instanton effects. It is well known that a positive root $+\alpha$ of the Lie algebra E_d corresponds to an instanton $\mathcal{B}_\alpha$ of M-theory compactified on T^d (see [46, 47] and section section 2.3 for a review). For example, for $d = 8$, if the metric on T^8 is diagonal and there are no fluxes, the instantons are Kaluza-Klein particles, M2-branes, M5-branes, and Kaluza-Klein monopoles with a euclidean world-volume. We will review this correspondence between positive roots and instantons in section 2.3.

In this paper we will study the case $d = 10$. This case is unique in that the Lie algebra E_{10} is the first E_d with a Cartan form that is not semi-positive definite. It is a hyperbolic Kac-Moody algebra with a Cartan form of signature $(9, 1)$. The roots α do not necessarily square to 2. In fact the roots of an infinite dimensional Kac-Moody Lie algebra can be classified as *real* and *imaginary* [41]. Real roots satisfy $\alpha^2 = 2$, and all other roots are called *imaginary* and satisfy $\alpha^2 \leq 0$. The Weyl group acts transitively on the real roots. We will review these facts in more detail in section 2.1.

The familiar instantons such as Kaluza-Klein particles, M2-branes, M5-branes, and Kaluza-Klein monopoles all correspond to *real* positive roots of E_{10}. In fact, as will be reviewed in section 2.3, the Weyl group of E_{10} formally acts as U-duality on the instanton [46, 47]. Hence, every object that can be obtained by U-duality from the above list of objects is also related to a positive real root, and, vice versa, every positive real root is related to an object that can be obtained by a formal U-duality transformation on, say, a euclidean M2-brane.

The question arises: *what is the physical interpretation of the imaginary roots?*

The purpose of this paper is to study the roots with $\alpha^2 \leq 0$ and to relate them to physical objects. We begin in section 3 by associating a formal "action" to the root, and we study the "combinatorial" properties of this action as a function of radii $R_1, \dots R_{10}$. In this section we explore a "naive" interpretation of imaginary roots simply as new types of instantons with very large actions.

In section 4 we propose a different interpretation, which is one of the main points of this paper. We propose that certain imaginary roots correspond to Minkowski objects. To support this claim, we construct the Minkowski objects — say branes – via a creation process by pushing one instanton through another. For example, one can construct an M2-brane by pushing an M5-brane through another M5-brane [48]. We use a Wick rotated version of that process where one instanton is translated in time until it crosses over another.

Once we accept the connection between imaginary roots and physical branes, we can study the interactions of branes with branes and the interactions of branes with fluxes from the Lie algebraic point of view. We characterize various interactions according to the inner product of the participating roots.

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Finally, we attempt to collect all the information together and construct an effective hamiltonian that describes the masses of the branes. The model is a σ-model on a coset G_{10}/K_{10} of G_{10}. The hamiltonian is, up to a sign, simply the $G_{10}(\equiv \exp E_{10})$ left-invariant laplacian $\mathcal{H} = -\triangle$ and the wave-function satisfies a Wheeler-DeWitt equation $\mathcal{H}\Psi = 0$. G_{10} is spontaneously broken to the U-duality subgroup $E_{10}(\mathbb{Z})$ by requiring Ψ to be $E_{10}(\mathbb{Z})$ invariant. This suggestion is rather old, but the new point is to try to analyze the modes that correspond to imaginary roots quantum mechanically. Doing that, we discover a piece in the hamiltonian that is analogous to a particle in a magnetic field. We compare the n^{th} excited Landau level to a state with n branes (or Kaluza-Klein particles). The energy separation between the Landau levels almost matches the energy of a brane, but unfortunately there is a mismatch by a factor of 2π. There are also a few other puzzles, related to charge neutrality and zero-point energies.

The paper is organized as follows. In section 2, we review the construction of infinite dimensional Lie algebras as presented in [41]. We also review billiard cosmology and the connection between M-theory on T^d and the Lie algebra E_d. In particular we discuss real and imaginary roots of E_{10} and their multiplicities. In section 3, we explore the combinatorial properties of branes that correspond to imaginary roots. In section 4, we argue that certain imaginary roots correspond to Minkowski branes and we study the various constructions of such branes via a brane creation process involving two instantons. As an application, in section 4.8, we add matter in the form of Kaluza-Klein particles and branes to billiard cosmology. The matter component corresponds to potentials in $(\log R_i)$-space oriented in directions corresponding to imaginary roots. In section 5, we study how interactions of pairs of branes and the interaction of a brane with a flux are encoded in the product of the corresponding roots. In section 6, we show that each instanton defines a subgroup of the maximally compact subgroup $K_{10} \subset \exp E_{10}$. This is an extension of the statement for $d = 8$ that a BPS instanton preserves half of the supersymmetry generators, and therefore defines a subgroup of the R-symmetry group Spin(16) $\subset E_{8(8)}(\mathbb{R})$. Finally in section 7, we explore a possible hamiltonian formulation. We conclude with some open questions and a few conjectures.

2. Preliminaries

2.1 Infinite dimensional Kac-Moody Lie algebras

In this subsection we will review the salient features of infinite dimensional Kac-Moody Lie algebras. Our discussion is taken from [49, 50, 41].

Readers who are familiar with this subject and readers who are not interested in the mathematical details are (reluctantly) advised to read section 2.1.1 and then skip to section 2.1.3. In section 2.1.1, we review the construction of Kac-Moody algebras, and demonstrate it for the hyperbolic Kac-Moody algebra of interest E_{10} and also for its subalgebra E_9 which is an example of an affine Lie algebra [51, 41]. In section 2.1.2, we explain the multiplicity formula for level $0, 1$ roots obtained in [50].

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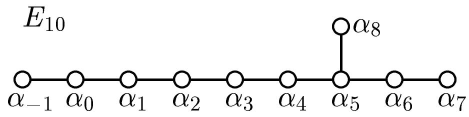
2.1.1 Review of Kac Moody algebras and root spaces

We recall the definition of the Kac-Moody algebra E_{10} and distinguished subalgebras E_8, E_9. The construction of E_{10} is a special case of the general construction of Kac Moody algebras in [49]. We start with the Dynkin diagram of E_{10}:



We then associate to the diagram a Cartan matrix $A_{10} = (a_{ij}), (i, j = -1, 0 \cdots 8)$, by defining

$$a_{ij} \stackrel{\text{def}}{=} \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if nodes } i, j \text{ are connected by a line} \\ 0 & \text{otherwise} \end{cases}$$

The matrix A_{10} is symmetric and $\det(A_{10}) = -1$; therefore $\text{rank}(A_{10}) = 10$. Choose a real vector space $\hat{\mathfrak{h}}_{\mathbb{R}}$ of dimension 10, and linearly independent sets $\Pi \stackrel{\text{def}}{=} \{\alpha_{-1}, \cdots, \alpha_8\} \subset \hat{\mathfrak{h}}^*_{\mathbb{R}}$ (where $*$ denotes the dual space) and $\Pi^\vee \stackrel{\text{def}}{=} \{\alpha^\vee_{-1}, \ldots, \alpha^\vee_8\} \subset \hat{\mathfrak{h}}_{\mathbb{R}}$ and define $\alpha_j(\alpha^\vee_i) \stackrel{\text{def}}{=} a_{ij}$. We note that for general Kac-Moody algebras

$$\dim \hat{\mathfrak{h}}_{\mathbb{R}} = 2n - \text{rank}(A) \tag{2.1}$$

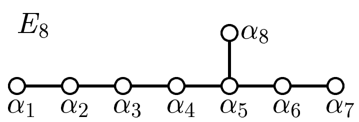
where n is the number of nodes in the Dynkin diagram and A is the matrix associated to the diagram.

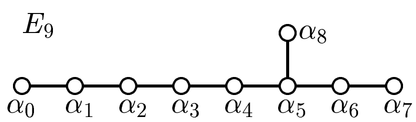
The Kac-Moody algebra E_{10} is the Lie algebra over $\mathbb{C}$ with the set of generators $\hat{\mathfrak{h}}_{\mathbb{R}} \cup \{e_i, f_i\}_{i=-1}^{8}$, and relations

$$\begin{aligned} &[h, h'] = 0, \quad [e_i, f_j] = \delta_{ij}\alpha^\vee_i, \quad [h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i\,, \qquad h, h' \in \hat{\mathfrak{h}}_{\mathbb{R}}, \\ &\text{ad}(e_i)^{1-a_{ij}}e_j = 0, \quad \text{ad}(f_i)^{1-a_{ij}}f_j = 0\,, \qquad i \neq j, \end{aligned} \tag{2.2}$$

where $\text{ad}(x)y \stackrel{\text{def}}{=} [x, y]$, $\text{ad}(x)^2y \equiv [x, [x, y]]$, and so on. Since $\hat{\mathfrak{h}}_{\mathbb{R}}$ has a basis of dimension 10, there are 30 linearly independent generators. These are called *Chevalley generators.* $\hat{\mathfrak{h}}_{\mathbb{R}}$ is called the *Cartan subalgebra* of E_{10} and is an abelian subalgebra of maximal dimension.

We next identify an E_9 subalgebra of E_{10} as the Kac Moody algebra obtained from the subdiagram of the E_{10} diagram by deleting the (-1)-node and the line connecting it to the 0-node. Similarly we identify an E_8 subalgebra by deleting the $-1, 0$ nodes and the lines connecting nodes $-1, 0$ and nodes $0, 1$.





We then construct corresponding Cartan matrices A_8 and A_9 following the procedure outlined above, and view these matrices as minors of A_{10}. The defining relations for E_8 and E_9 are thus inherited from the relations for E_{10}.

We let $\dot{\mathfrak{h}}_{\mathbb{R}}$ denote the Cartan subalgebra (CSA) of E_8 and $\mathfrak{h}_{\mathbb{R}}$ the CSA of E_9. We note that $\det(A_8) = 1$ and $\det(A_9) = 0$. A basis for the kernel of A_9 is $\{(0, 1, 2, 3, 4, 5, 6, 4, 2, 3)^t\}$.

We then see from the adaptation of formula (2.1) to E_9 that $\dim \mathfrak{h}_{\mathbb{R}} = 10$, and thus $\mathfrak{h}_{\mathbb{R}} = \hat{\mathfrak{h}}_{\mathbb{R}}$. In keeping with the notation of [50], we define $\dot{\mathbf{g}} \stackrel{\text{def}}{=} E_8$ with CSA $\dot{\mathfrak{h}}_{\mathbb{R}}$, $\mathbf{g} \stackrel{\text{def}}{=} E_9$ with CSA $\mathfrak{h}_{\mathbb{R}}$, and $\hat{\mathbf{g}} \stackrel{\text{def}}{=} E_{10}$ with CSA $\hat{\mathfrak{h}}_{\mathbb{R}}$. We have the root space decompositions of each algebra with respect to its CSA. For example, $\hat{\mathbf{g}} = \bigoplus_{\alpha \in \hat{\mathfrak{h}}^*_{\mathbb{R}}} \hat{\mathbf{g}}_\alpha$ where

$$\hat{\mathbf{g}}_\alpha \stackrel{\text{def}}{=} \{x \in \hat{\mathbf{g}} : [h, x] = \alpha(h)x\ ,\ \ \forall h \in \hat{\mathfrak{h}}_{\mathbb{R}}\}\,,$$

and we define the *root space*

$$\hat{\Delta} \stackrel{\text{def}}{=} \{\alpha \in \hat{\mathfrak{h}}^*_{\mathbb{R}} : \hat{\mathbf{g}}_\alpha \neq 0, \alpha \neq 0\}\,.$$

We let $\hat{Q} \stackrel{\text{def}}{=} \sum_{i=-1}^{8} \mathbb{Z}\alpha_i$, and $\hat{Q}_+ \stackrel{\text{def}}{=} \sum_{i=-1}^{8} \mathbb{N}\alpha_i$. ($\mathbb{N}$ will denote the non-negative integers.) Finally, define $\hat{\Delta}_+ = \hat{\Delta} \cap \hat{Q}_+$, the set of positive roots of $\hat{\mathbf{g}}$. We then have $\hat{\Delta} = \hat{\Delta}_+ \cup \hat{\Delta}_-$ where $\hat{\Delta}_- = -\hat{\Delta}_+$ [41]. We define $\dot{Q} \subset \hat{Q}$, $\dot{\Delta} \subset \hat{\Delta}$, etc., analogously for the algebras $\dot{\mathbf{g}}$, and similarly for $\mathbf{g}$.

The signature of the inner product on the root lattice of a finite simple Lie algebra is well known to be positive definite [41], so from the E_8 subalgebra of E_{10} and the fact that $\det(A_{10}) = -1$, we see that the inner product on $\hat{Q}$ must have signature $(9, 1)$.

We partial-order $\hat{\mathfrak{h}}^*_{\mathbb{R}}$ by $\alpha \succeq \beta$ if $\alpha - \beta \in \hat{Q}_+$. For $\alpha = \sum_{i=-1}^{8} k_i\alpha_i \in \hat{Q}$ we define the *height* as $\mathrm{ht}(\alpha) \stackrel{\text{def}}{=} \sum_{i=-1}^{8} k_i$. Finally, we introduce the *Weyl group* $\hat{W}$ of $\hat{\mathbf{g}}$ as the subgroup of $\mathrm{Aut}\,\hat{\mathfrak{h}}^*_{\mathbb{R}}$ (the group of metric preserving linear transformations of $\hat{\mathfrak{h}}_{\mathbb{R}}$) generated by *simple reflections*

$$r_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i\,, \qquad i = -1, \ldots, 8\,, \qquad \lambda \in \hat{\mathfrak{h}}^*_{\mathbb{R}}\,.$$

A root $\alpha \in \hat{\Delta}$ is called a *real root* if there exist $w \in \hat{W}$ such that $w(\alpha) = \alpha_i$ for some $-1 \leq i \leq 8$; otherwise α is an *imaginary root.* As $\dot{\mathbf{g}}$ is a finite dimensional Lie algebra, all of its roots are real. In general, a root α is real if and only if $(\alpha|\alpha) > 0$. For $E_9 \equiv \mathbf{g}$, all the imaginary roots are integer multiples of the root

$$\delta \stackrel{\text{def}}{=} \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 \in \Delta\,. \tag{2.3}$$

It satisfies

$$(\delta|\alpha_i) = 0\,, \qquad i = 0, \ldots, 8\,,$$

We denote the set of imaginary roots of E_{10} as

$$\hat{\Delta}_{\text{Im}} \stackrel{\text{def}}{=} \{\alpha \in \hat{\Delta} : (\alpha|\alpha) \leq 0\}\,,$$

and we define the set of positive (negative) imaginary roots as $\hat{\Delta}^+_{\text{Im}} \stackrel{\text{def}}{=} \hat{\Delta}_{\text{Im}} \cap \hat{\Delta}_+$ ($\hat{\Delta}^-_{\text{Im}} \stackrel{\text{def}}{=} \hat{\Delta}_{\text{Im}} \cap \hat{\Delta}_-$).

The adjoint action of $\hat{\mathbf{g}}$ on itself is an integrable representation, which means that

$$\forall x \in \hat{\mathbf{g}} \qquad \exists n \in \mathbb{Z}_+ : \quad \mathrm{ad}(e_{\alpha_i})^n(x) = 0\,, \quad \mathrm{ad}(f_{\alpha_i})^n(x) = 0\,, \qquad i = -1, \ldots, 8$$

Among other things, it implies that the Lie group $\exp \hat{\mathbf{g}}$ can be defined. It also implies that $\hat{W}$ preserves multiplicities of roots. Therefore, all real roots have multiplicity 1. However,

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imaginary roots have multiplicities greater than 1. The multiplicities of the imaginary roots of $\mathbf{g}$ are given by

$$\mathrm{mult}(n\delta) = 8\,, \qquad 0 \neq n \in \mathbb{Z}\,.$$

There is no known closed formula for the multiplicities of the imaginary roots of $\hat{\mathbf{g}} \equiv E_{10}$. However, a closed formula has been derived in [50] for roots of "affine levels" 1 and 2 (this term will be explained below). We outline the derivation of these multiplicities in the next subsection. See also [36, 38] for a list of many roots and their multiplicities.

We will make a few extra observations before we continue.

Proposition 1 (Lemma 5.3 and Theorem 5.4 of [41]). *Every imaginary root can be uniquely written as $\gamma = w(\alpha)$ for a Weyl-group element $w \in \hat{W}$ and $\alpha \equiv \sum_{i=-1}^{8} k_i\alpha_i \in \hat{Q}$ satisfying: (i) $(\alpha|\alpha_i) \leq 0$ for all simple roots ($i = -1, \ldots, 8$), and (ii) the subdiagram of the Dynkin diagram consisting of all vertices such that $k_i \neq 0$ is connected.*

Proposition 2. *Every imaginary root $\alpha \in \hat{\Delta}_{\mathrm{Im}}$ that satisfies $(\alpha|\alpha) = 0$ is $\hat{W}$-equivalent to $n\delta$ for some $0 \neq n \in \mathbb{Z}$. Its multiplicity is therefore exactly* 8.

Proof. This follows immediately from proposition 5.7 of [41], which uses proposition 1. □

Proposition 3. *Every positive imaginary root $\alpha \in \hat{\Delta}_{\mathrm{Im}}$ that satisfies $(\alpha|\alpha) = -2$ is $\hat{W}$-equivalent to $\alpha_{-1} + 2\alpha_0 + 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 10\alpha_4 + 12\alpha_5 + 8\alpha_6 + 4\alpha_7 + 6\alpha_8$.*

Proof. We use the same technique as in the proof of proposition 5.7 of [41]. We set $\alpha = \sum_{i=-1}^{8} k_i\alpha_i$ with $k_i \geq 1$ (otherwise $\alpha^2 \geq 0$). Using proposition 1, we may assume that $(\alpha|\alpha_i) \leq 0$ for all $i = -1, \ldots, 8$. Then $-2 = (\alpha|\alpha) = \sum_{i=-1}^{8} k_i(\alpha|\alpha_i)$. But every term on the righthand side is negative or zero. Since all k_i's are positive we are left with three options: (i) $(\alpha|\alpha_s) = (\alpha|\alpha_t) = -1$ for some $-1 \leq s < t \leq 8$, and $(\alpha|\alpha_i) = 0$ for all $i \neq s, t$; (ii) $(\alpha|\alpha_s) = -2$ and $k_s = 1$ for some $-1 \leq s \leq 8$, and $(\alpha|\alpha_i) = 0$ for all $i \neq s$; (iii) $(\alpha|\alpha_s) = -1$ and $k_s = 2$ for some $-1 \leq s \leq 8$, and $(\alpha|\alpha_i) = 0$ for all $i \neq s$.

Using the inverse of the Cartan matrix given in (2.6) below, we can solve all k_i's in each case above, and check whether $\alpha^2 = -2$. It turns out that there is a unique solution, and only for case (iii) with $s = 2$, which is the root given above. □

As we shall see in section 2.1.2, the multiplicity of the root is 44. Therefore, all roots α with $\alpha^2 = -2$ have multiplicity 44.

Definition 1. We will say that a root α is *prime* if it cannot be written as $\alpha = n\beta$ for some integer $n > 1$ and a root β.

All real roots are prime, but imaginary roots are not necessarily prime. Since all roots with $(\alpha|\alpha) = 0$ are Weyl-equivalent to a multiple of the root δ, it follows that all positive prime roots with $(\alpha|\alpha) = 0$ are Weyl equivalent to the root δ.

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Let us summarize the various terms in the following table:

$\hat{\mathfrak{h}}_{\mathbb{R}}$	Cartan subalgebra of E_{10}
$\hat{\Delta}$	Set of all roots of E_{10}
$\hat{\Delta}_+$	Set of positive roots of E_{10}
$\hat{\Delta}_{\mathrm{Re}}$	Set of real roots ($\alpha^2 = 2$) of E_{10}
$\hat{\Delta}_{\mathrm{Im}}$	Set of imaginary roots ($\alpha^2 \leq 0$) of E_{10}
$\hat{Q}$	Root lattice of E_{10}
$\hat{W}$	Weyl group of E_{10}
$\succeq$	partial order on $\hat{\mathfrak{h}}^*_{\mathbb{R}}$
ht	height of a root
$\mathfrak{h}_{\mathbb{R}}$	Cartan subalgebra of E_9
Δ	Set of all roots of E_9
$\vdots$	
δ	minimal positive imaginary root of E_9
$\dot{\mathfrak{h}}_{\mathbb{R}}$	Cartan subalgebra of E_8
$\dot{\Delta}$	Set of all roots of E_8
$\vdots$	

2.1.2 Dimensions of Level-1 root spaces

For an element $\alpha = \sum_{i=-1}^{8} k_i\alpha_i \in \hat{Q}$, $-k_0 = (\alpha|\delta)$ is called the *affine level* of α. Here $\delta \in \Delta \subset \hat{\Delta}$ was defined in (2.3). We denote the set of all roots of E_{10} at affine level l by $\hat{\Delta}_{[l]}$.

The formula

$$\mathrm{mult}(\alpha) = p^{(8)}\left(1 - \frac{(\alpha|\alpha)}{2}\right), \qquad -k_0 = 0, 1\,.$$

is derived in [50] for α a level 0 or level 1 root of E_{10}. By definition, $p^{(8)}(k)$ is the coefficient of q^k in $1/\prod_{n=1}^{\infty}(1-q^n)^8$. Up to numerical prefactors, the generating function of bosonic objects $\prod_{n=1}^{\infty}[1/(1-q^n)]$ is ubiquitous in string theory, and its appearance in this new context is very intriguing.

The derivation in [50] makes reference to chapter 12 in [41], and we briefly fill in those details here. Define the *weights* of E_{10} as

$$\hat{P} \stackrel{\mathrm{def}}{=} \{\lambda \in \hat{\mathfrak{h}}^*_{\mathbb{R}} : (\lambda|\alpha_i) \in \mathbb{Z}\,,\ i = -1, 0, \ldots, 8\}\,,$$

and define the *dominant weights* as

$$\hat{P}_+ \stackrel{\mathrm{def}}{=} \{\lambda \in \hat{P} : (\lambda|\alpha_i) \geq 0,\ i = -1, 0, \ldots, 8\} \tag{2.4}$$

In [50], the *dominant weights* in the weight lattice $\hat{Q}$ are defined in a different way:

$$\hat{P}_+ = \{\sum_{i=-1}^{8} k_i\hat{\Lambda}_i : k_i \in \mathbb{N}\}$$

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where $\hat{\Lambda}_i$ $(i = -1, 0, \ldots, 8)$ are the *fundamental weights*,

$$(\hat{\Lambda}_i|\alpha_j) = \delta_{i,j}\,, \qquad i, j = -1, 0, \ldots, 8. \tag{2.5}$$

The two definitions are equivalent. It is obvious that $\hat{P} \supseteq \hat{Q}$, which is true for any Kac-Moody algebra. For E_{10}, since $\det(A_{10}) = -1$, it follows that $\hat{P} = \hat{Q}$.

The fundamental weights are calculated as follows [50]. Expand $\hat{\Lambda}_i = \sum_{k=-1}^{8} c_{ik}\alpha_k$. Then we solve

$$\sum_{k=-1}^{8} c_{ik}(\alpha_k|\alpha_j) = \sum_{k=-1}^{8} c_{ik}a_{kj} = \delta_{i,j}\,.$$

Thus, the coefficients c_i are the rows of the inverse of the Cartan matrix $A_{10} = (a_{ij})$:

$$-(A_{10})^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 1 & 2 & 4 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 2 & 4 & 6 & 9 & 12 & 15 & 18 & 12 & 6 & 9 \\ 3 & 6 & 9 & 12 & 16 & 20 & 24 & 16 & 8 & 12 \\ 4 & 8 & 12 & 16 & 20 & 25 & 30 & 20 & 10 & 15 \\ 5 & 10 & 15 & 20 & 25 & 30 & 36 & 24 & 12 & 18 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 28 & 14 & 21 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 18 & 9 & 14 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 9 & 4 & 7 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 14 & 7 & 10 \end{pmatrix} \tag{2.6}$$

We have

$$\hat{\Lambda}_{-1} = -\delta\,, \qquad \hat{\Lambda}_0 = -\alpha_{-1} - 2\delta\,.$$

From [41] (chapter 5) we know that the set of negative imaginary roots $\hat{\Delta}^-_{\mathrm{Im}}$ is $\hat{W}$-invariant. The orbit of $\hat{W}$ on an imaginary root of $\hat{\mathbf{g}} = E_{10}$ intersects $\hat{P}_+$ exactly once; the intersection root μ is the one that maximizes $\mathrm{ht}(\mu)$ [41] (chapter 5). Since $\hat{\mathbf{g}}$ is integrable as an adjoint representation of itself, the Weyl group $\hat{W}$ preserves root multiplicities, so it suffices to find the multiplicities of $\hat{\Delta}^-_{\mathrm{Im}} \cap \hat{P}_+$. It is easily checked from the second definition of $\hat{P}_+$, given above (2.5), that dominant weights that are also roots at level-1 are of the form

$$\hat{\Delta}_{[1]} \cap \hat{P}_+ = \{\hat{\Lambda}_0 + k_{-1}\hat{\Lambda}_{-1} = -\alpha_{-1} - (k_{-1}+2)\delta : k_{-1} \in \mathbb{N}\}\,.$$

The idea in [50] is to determine the multiplicities of these level-1 roots.

Given $\hat{\Lambda} \in \hat{\mathfrak{h}}_{\mathbb{R}}$, denote by $L(\hat{\Lambda})$ the irreducible representation of E_9 with highest weight $\hat{\Lambda}$ (chapter 9 of [41]). $L(\hat{\Lambda})$ has weight space decomposition $L(\hat{\Lambda}) = \bigoplus_{\lambda \leq \hat{\Lambda}} V_\lambda$, where $\dim(V_{\hat{\Lambda}}) = 1$. $L(\hat{\Lambda})$ is integrable if and only if $\hat{\Lambda} \in \hat{P}_+$ (Chapter 10 of [41]).

Note that $\hat{\Delta}_{[1]}$, defined at the beginning of this subsection, is a representation of $\mathbf{g} \equiv E_9$. From now till the rest of this subsection we restrict attention to this representation.

We note that $-\alpha_{-1} = \hat{\Lambda}_0 + 2\delta$, and $L(\hat{\Lambda}_0 + 2\delta)$ is an integrable highest weight representation of E_9. The level of the representation, as a representation of an affine Lie algebra, is $(-\alpha_{-1}|\delta) = 1$. In general, let $P(\Lambda)$ be the set of weights of a representation $L(\Lambda)$ of

$\mathbf{g} \equiv E_9$, with $\Lambda \in P_+$, where P, P_+ are defined as in (2.4) but for E_9:

$$P \stackrel{\text{def}}{=} \{\lambda \in \mathfrak{h}_{\mathbb{R}}{}^* : (\lambda|\alpha_i) \in \mathbb{Z},\ i = 0, \ldots, 8\},$$
$$P_+ \stackrel{\text{def}}{=} \{\lambda \in P : (\lambda|\alpha_i) \geq 0,\ i = 0, \ldots, 8\}.$$

Then $\lambda \in P(\Lambda)$ is called *maximal* if $\lambda + \delta \notin P(\Lambda)$. We denote the set of maximal weights of $L(\Lambda)$ by

$$\mathrm{Max}(\Lambda) \stackrel{\text{def}}{=} \{\lambda \in P(\Lambda) : \lambda + \delta \notin P(\Lambda)\}$$

Claim 1. $\mathrm{Max}(\Lambda)$ *is preserved by the Weyl group* W *of* $\mathbf{g} \equiv E_9$.

Proof. Suppose $w(\lambda) + \delta \in P(\Lambda)$ for some $\lambda \in \mathrm{Max}(\Lambda)$ and $w \in W$. Then $w^{-1}(w(\lambda) + \delta) \in P(\Lambda)$. But $\lambda + w^{-1}(\delta) = \lambda + \delta$, so we have a contradiction. □

Any orbit of W on $P(\Lambda)$ intersects P_+ once; the intersection weight μ being the weight such that $\mathrm{ht}(\Lambda - \mu)$ is minimal in its W orbit. In particular, any maximal weight is W-equivalent to a maximal weight in P_+. Since $L(\hat{\Lambda}_0 + 2\delta = -\alpha_{-1})$ is highest weight, $-\alpha_{-1} + \delta$ is not in $P(\hat{\Lambda}_0 + 2\delta)$; therefore $\hat{\Lambda}_0 + 2\delta$ is a maximal weight in $P(\hat{\Lambda}_0 + 2\delta) \cap P_+$. It is the unique such weight [50]. From previous remarks it then follows that any weight of $\mathrm{Max}(\Lambda_0 + 2\delta)$ is W-equivalent to $\hat{\Lambda}_0 + 2\delta$. We now state

Proposition 4 **(**12.5(e) **of [41]).** *For any* $\mu \in P(\Lambda)$*, there exists a unique* $\lambda \in \mathrm{Max}(\Lambda)$ *and unique* $n \geq 0$ *such that* $\mu = \lambda - n\delta$*. Furthermore, for* $\lambda \in P(\Lambda)$*, the set* $\{n \in \mathbb{Z} : \lambda - n\delta \in P(\Lambda)\}$ *is an interval* $[-p, \infty)$ *with* $p \geq 0$*, and the function* $t \mapsto \mathrm{mult}_{L(\Lambda)}(\lambda - t\delta)$ *is non-decreasing on the interval. Moreover, if* $0 \neq x \in \mathbf{g}_{-\delta}$*, (where* $\mathbf{g}_{-\delta} \subset \mathbf{g}$ *is the* 8*-dimensional subspace of the Lie algebra* E_9 *of all the elements* $x \in \mathbf{g}$ *with weight* $-\delta$*) the map* $\mathrm{ad}(x) : L(\Lambda)_{\lambda - t\delta} \to L(\Lambda)_{\lambda-(t+1)\delta}$ *given by* $y \mapsto [x, y]$ *is injective.*

These observations imply

$$P(\Lambda) = \bigsqcup_{\lambda \in \mathrm{Max}(\Lambda)} \{\lambda - n\delta : n \geq 0\}. \tag{2.7}$$

(The union is disjoint.)

We will now define a few characters. The expressions below are *formal* series in the formal variables e^μ where μ runs over all possible weights. They are of the form $\sum_\mu k_\mu e^\mu$ where k_μ are integers. Two such series can be multiplied to yield a series of a similar form, and the integer multiplicities k_μ can be read off the coefficient of e^μ. (There are actually some restrictions on multiplying two series — it is required that each resulting k_μ will have a finite number of contributions, but we do not need to worry about that here.) We will also use the convention that $(1 - e^{-\mu})^{-1} = 1 + e^{-\mu} + e^{-2\mu} + \cdots$.

First, for $\lambda \in \mathrm{Max}(\Lambda)$, define

$$a_\lambda^\Lambda \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \mathrm{mult}_{L(\Lambda)}(\lambda - n\delta) e^{-n\delta}.$$

Also, define the character $\operatorname{Ch} L(\Lambda)$ of $P(\Lambda)$ as

$$\operatorname{Ch} L(\Lambda) \stackrel{\text{def}}{=} \sum_{\lambda\in P(\Lambda)} (\dim_{L(\Lambda)} \lambda) e^{\lambda} .$$

The above decomposition (2.7) of $P(\Lambda)$ implies that

$$\operatorname{Ch} L(\Lambda) = \sum_{\lambda\in \operatorname{Max}(\Lambda)} e^{\lambda} a^{\Lambda}_{\lambda} .$$

We now return to the level-1 representation of interest, $L(\hat{\Lambda}_0 + 2\delta)$. We proved above that any W-orbit in $\operatorname{Max}(\Lambda)$ intersects P_+ exactly once. Since $P_+ \cap \hat{\Delta}_{[1]} = \hat{\Lambda}_0 + 2\delta$, there is therefore only one W orbit. The character $\operatorname{Ch} L(\hat{\Lambda}_0 + 2\delta)$ therefore contains the term

$$e^{\hat{\Lambda}_0+2\delta} a^{\hat{\Lambda}_0+2\delta}_{\hat{\Lambda}_0+2\delta} ,$$

and a term with the same root multiplicities and the same values of $(\alpha|\alpha)$ for each maximal weight that is W-equivalent to $\hat{\Lambda}_0 + 2\delta$. To proceed, we quote

Proposition 5 (12.13 of [41]). *Let $\Lambda \in P^1_+ \stackrel{def}{=} P_+ \cap \hat{\Delta}_{[1]}$. Then (and, by the way, this is true in general for affine algebras of type $X^{(r)}_N$, where $X = A, D$ or E),*

$$a^{\Lambda}_{\Lambda} = \prod_{n=1}^{\infty} (1 - e^{-n\delta})^{-\operatorname{mult}(n\delta)} .$$

Recalling the realization of the affine algebra $\mathbf{g} \equiv E_9$ as a Lie algebra of regular polynomial maps from $\mathbb{C}^*$ to $\dot{\mathbf{g}} \equiv E_8$, we know that $\dim(\mathbf{g}_{n\delta}) = \dim(\dot{\mathfrak{h}}_{\mathbb{R}} \otimes t^n) = 8$. Since $\hat{\Lambda}_0 + 2\delta \in P^1_+$, the above observation implies a term

$$e^{\hat{\Lambda}_0+2\delta} \prod_{n=1}^{\infty} (1 - e^{-n\delta})^{-8}$$

in the character $\operatorname{Ch} L(\hat{\Lambda}_0 + 2\delta)$. Let $\alpha = \hat{\Lambda}_0 + 2\delta - k\delta = -\alpha_{-1} - k\delta$. We have

$$(\alpha|\alpha) = 2(1-k) \Rightarrow k = 1 - \frac{(\alpha|\alpha)}{2} .$$

Define $p^{(8)}(k)$ to be the coefficient of $e^{-k\delta}$ in $\prod_{n=1}^{\infty}(1-e^{-n\delta})^{-8}$, and we have that $\operatorname{mult}(\alpha) = p^{(8)}(k)$. Putting this together gives Kac's result

$$\operatorname{mult}(\alpha) = p^{(8)}\big(1 - \frac{(\alpha|\alpha)}{2}\big)$$

for α a level-0 or level-1 root.

2.1.3 "Physical" basis for the Cartan subalgebra of E_{10}

It is convenient to pick a basis for the Cartan subalgebra of E_{10} that exhibits the $sl(10) \subset E_{10}$ subalgebra manifestly. In this basis a vector $\vec{h} \in \hat{\mathfrak{h}}_{\mathbb{R}}$ has components

$$\vec{h} = (h_1, h_2, \ldots, h_{10})\,. \tag{2.8}$$

The relation to the basis $\alpha_{-1}, \ldots, \alpha_8$ of section 2.1.1 is given by

$$\vec{h} = \sum_{i=-1}^{5}\left(\sum_{j=1}^{i+2} h_j\right)\alpha_i^\vee + \frac{1}{3}\left(2\sum_{j=1}^{8} h_j - h_9 - h_{10}\right)\alpha_6^\vee + \frac{1}{3}\left(\sum_{j=1}^{9} h_j - 2h_{10}\right)\alpha_7^\vee + \frac{1}{3}\left(\sum_{j=1}^{10} h_j\right)\alpha_8^\vee\,. \tag{2.9}$$

Acting as a subgroup of the Weyl group $\hat{W}$ of E_{10}, the Weyl group of $sl(10)$, which is the permutation group S_{10} simply permutes the components $h_1, \ldots, h_{10}$. The Cartan metric can be written in this basis as

$$\|\vec{h}\|^2 = \sum_{i=1}^{10} h_i^2 - \left(\sum_{i=1}^{10} h_i\right)^2\,. \tag{2.10}$$

Similarly, we define a "physical" basis for $\hat{\mathfrak{h}}^*_{\mathbb{R}}$ as follows:

$$\begin{aligned}
\alpha_{-1} &= (1,-1,0,0,0,0,0,0,0,0)\,,\\
\alpha_0 &= (0,1,-1,0,0,0,0,0,0,0)\,,\\
&\;\vdots\\
\alpha_7 &= (0,0,0,0,0,0,0,0,1,-1)\,,\\
\alpha_8 &= (0,0,0,0,0,0,0,1,1,1)\,,
\end{aligned}$$

For any root α, we will frequently use the notation

$$\alpha^2 \stackrel{\text{def}}{=} (\alpha|\alpha)\,.$$

2.2 Billiard cosmology

We will now review the classical evolution of a universe constructed from M-theory on T^{10}. We used the term "constructed from" because, as we shall see, inclusion of generalized fluxes can change the topology. Our review is based in part on [9] and [28].

The initial ansatz is a Kasner-like metric

$$ds^2 = -dt^2 + \sum_{i=1}^{10} R_i(t)^2 dx_i^2\,, \qquad 0 \le x_i < 2\pi\,, \qquad i = 1, \ldots, 10\,. \tag{2.11}$$

Einstein's equations are solved by

$$\log\frac{R_i(t)}{R_i(t_0)} = p_i \log\frac{t}{t_0}\,, \qquad i = 1, \ldots, 10\,,$$

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(for some fixed arbitrary t_0) provided that

$$\sum_{i=1}^{10} p_i = \sum_{i=1}^{10} p_i^2 = 1\,. \tag{2.12}$$

For fixed

$$\tau \equiv \log\frac{t}{t_0}\,,$$

define the ten-dimensional vector

$$\vec{h} \equiv (\log[M_p R_1], \dots, \log[M_p R_{10}])\,. \tag{2.13}$$

It is convenient to interpret this vector as a point in the Cartan subalgebra $\hat{\mathfrak{h}}_{\mathbb{R}} \subset E_{10}$ according to (2.8). The classical evolution of the universe is now mapped to an abstract mechanical system of a single particle moving on a straight line in $\hat{\mathfrak{h}}_{\mathbb{R}}$. If we identify $\tau = \log(t/t_0)$ as the time variable then the particle has constant velocity. Note that, with the Cartan metric (2.10), the configuration space $\hat{\mathfrak{h}}_{\mathbb{R}}$ is identified with $\mathbb{R}^{9,1}$.

Note that (2.12) implies that at least one p_i has to be negative and at least one other p_j has to be positive. This means that in the far past and in the far future at least one dimension shrinks to zero, according to the classical solution. This observation invalidates the assumptions of classical 10+1D geometry both in the far past and in the far future. As shown in [9], it is still possible to have a weakly coupled description after dimensional reduction, provided that $d\vec{h}/d\tilde{\tau}$ is timelike in the Cartan metric (2.10). This will not be the case if equation (2.12) is satisfied, but it could be true if we add matter. But first we include fluxes.

Denote the 4-form field strength of 10+1D supergravity by $G = dC$. To start, suppose we turn on only the component G_{1234}. Flux quantization requires it to be an integer. It then contributes a potential term to the classical supergravity action proportional to

$$\sqrt{g}|G|^2 = \frac{G_{1234}^2}{(R_1R_2R_3R_4)^2}V_{10} = \frac{G_{1234}^2}{V_{10}}(R_5\cdots R_{10})^2\,, \qquad V_{10} \equiv R_1\cdots R_{10}\,.$$

Note that in the absence of fluxes, the condition (2.12) implies that

$$d\tau = \frac{dt}{t} = C\frac{dt}{M_p^{10}V_{10}}\,, \qquad (G=0)\,.$$

where C is a constant. In the presence of fluxes, it is more convenient to define *conformal time* as

$$\tilde{\tau} \equiv \int_{t_0}^{t} \frac{dt'}{2\pi M_p^9 V_{10}(t')} \tag{2.14}$$

for some initial time t_0. It then turns out that the classical equations of motion are encoded by the lagrangian

$$L = 2\pi\left\|\frac{d\vec{h}}{d\tilde{\tau}}\right\|^2 - \pi[(2\pi)^3 G_{1234}]^2 e^{2(h_5+h_6+h_7+h_8+h_9+h_{10})} \tag{2.15}$$

with the extra constraint that only trajectories with total energy zero (defined with respect to the conformal time) are allowed. The potential term that is proportional to the square of the flux G_{1234} can be modeled as a sharp wall at position

$$h_5 + \cdots + h_{10} \sim -\log G_{1234}\,.$$

The mechanical system is now described by a particle moving at constant velocity (with respect to the conformal time $\tilde{\tau}$) until it hits the wall. After the collision the particle reflects off the wall, conserving energy and momentum parallel to the wall, and continues at a constant velocity on its new trajectory. It turns out that the reflection off the wall can be interpreted as a Weyl reflection in $\hat{\mathfrak{h}}_{\mathbb{R}}$. That is, the reflection off the wall defines a linear transformation on the velocity vector $d\vec{h}/d\tilde{\tau}$, which is precisely a Weyl reflection.

U-duality [5] acts on the vector $\vec{h}$. In fact the U-duality group has a subgroup that is generated by permutations of the indices $h_1, \ldots, h_{10}$ and by the transformation

$$\vec{h} \longrightarrow \left(h_1 - \frac{2}{3}h_{123}, h_2 - \frac{2}{3}h_{123}, h_3 - \frac{2}{3}h_{123}, h_4 + \frac{1}{3}h_{123}, \ldots, h_{10} + \frac{1}{3}h_{123}\right), \quad h_{123} \equiv h_1 + h_2 + h_3\,.$$

(The remaining U-duality transformation generators are transformations that enforce periodicity of gauge fluxes such as C_{123}.) It turns out that this subgroup is the Weyl group of E_{10} [46, 47]. These linear transformations preserve the kinetic term of (2.15), since the Weyl group preserves the Cartan metric. But they can act nontrivially on the potential term. As we have seen above, each potential term corresponds to a Weyl reflection in $\hat{\mathfrak{h}}_{\mathbb{R}}$. The Weyl group acts on these reflections by conjugation, and hence changes the position of the walls.

Some of the new walls obtained this way correspond to other fluxes, while other walls correspond to a topology change, because U-duality can turn the components G_{1234} into components of the metric. For example, one can get a wall that corresponds to the potential term

$$\pi k^2 \exp\{2(h_1 + h_2 + h_3 + h_4 + h_5 + h_6 + h_7 + 2h_{10})\}\,.$$

This wall describes a topology change from T^{10} to a circle fibration of the 10^{th} direction over the 8^{th} and 9^{th} with first Chern class $c_1 = k$. The metric is given by

$$ds^2 = -dt^2 + \sum_{i=1}^{9} R_i(t)^2 dx_i^2 + R_{10}(t)^2 \left(dx_{10} - \frac{k}{2\pi} x_9 dx_8\right)^2\,,$$

and the boundary conditions are such that $x_9 \to x_9 + 2\pi$ must be accompanied by $x_{10} \to x_{10} + kx_8$.

Finally, consider two fluxes in transverse directions, say G_{1234} and G_{5678}. The term $\int C \wedge G \wedge G$ of 10+1D supergravity implies that $G \wedge G$ is a source of 3-form flux. Since all 10 spatial dimensions are compact an anti M2-brane must be present to absorb the flux [52]. The Kasner cosmology must now also contain matter in addition to fluxes.

2.3 Instantons and positive roots

We have mentioned in section 2.2 that fluxes such as G_{1234} correspond to real positive roots of E_{10}. We will now discuss this correspondence in more detail. Instead of discussing the fluxes themselves, it is convenient to discuss processes that change the flux by one unit. These are the *instantons* of M-theory. For example, the flux G_{1234} can be changed by one unit via an instanton that can be interpreted as an M5-brane with euclidean world-volume, wrapping the $5^{th},\ldots,10^{th}$ directions [53, 54]. Analogous euclidean branes can also be constructed in string theory and supergravity. In this context they are known as S-branes [55]–[56].

Let us list the various possible euclidean objects present for M-theory on T^d with $d \leq 8$. These are: Kaluza-Klein particles, M2-branes, M5-branes, and Kaluza-Klein monopoles. Let $R_1,\ldots,R_d$ be the radii of T^d and M_p be the Planck mass. In the absence of fluxes, the actions of these objects are, up to permutations of the indices,

$$2\pi R_1 R_2^{-1}\,, \quad 2\pi M_p^3 R_1R_2R_3\,, \quad 2\pi M_p^6 R_1R_2R_3R_4R_5R_6\,, \quad 2\pi M_p^9 R_1R_2R_3R_4R_5R_6R_7R_8^2\,. \tag{2.16}$$

The correspondence with positive roots of E_8 allows us to write down a simple formula for such actions. Up to a 2π factor, the log of the action S_α of instanton $\mathcal{B}_\alpha$ is given by $\langle\alpha,\vec{h}\rangle$ where $\vec{h}$ is the vector in the Cartan subalgebra $\hat{\mathfrak{h}}_{\mathbb{R}}$ that is related to $R_1,\ldots,R_d$ by $\vec{h} = (\log[M_pR_1],\ldots,\log[M_pR_d])$, similarly to equation (2.13).

$$S_\alpha = 2\pi e^{\langle\alpha,\vec{h}\rangle}\,.$$

If $\vec{h}$ is in a region of $\hat{\mathfrak{h}}_{\mathbb{R}}$ such that $\langle\alpha_i,\vec{h}\rangle \gg 1$ for all simple roots $i=-1,\ldots,8$ then the euclidean objects can be safely interpreted as instantons. Generically, if the R_i's are given by (1.2) with $\vec{p}$ timelike in the metric (1.3) then there is some choice of simple roots for E_{10} for which all the instanton actions above are large at very late times [9].

The euclidean objects contribute instanton terms to amplitudes. These instanton terms could, for example, be corrections to R^4 terms (contractions of 4 curvature tensors) or λ^{16} terms (contractions of 16 fermions) in the low-energy effective action in the $(11-d)$ noncompact dimensions [47]. The instanton terms behave as $\Phi = \exp(-S_\alpha + i\mathcal{C}_\alpha)$, where $\mathcal{C}_\alpha$ is the flux that couples to the object. For example: for the Kaluza-Klein particle with action $S_\alpha = 2\pi R_1/R_2$, this flux is the ratio of metric components $\mathcal{C}_\alpha = 2\pi g_{12}/g_{22}$, for the M2-brane with action $S_\alpha = 2\pi M_p^3R_1R_2R_3$, the flux is the M-theory 3-form component $\mathcal{C}_\alpha = (2\pi)^3 C_{123}$.

Strictly speaking, the instanton actions in (2.16) are in the absence of off-diagonal metric components such as $g_{12},\ldots,$ and in the absence of 3-form fluxes such as C_{123}, etc. In order to avoid confusion with the 4-form flux $G = dC$ we will refer to all the former collectively as *θ-angles.* In the presence of θ-angles, the action S_α is, in general, modified. All the θ-angles, together with the radii $R_1,\ldots,R_d$ parameterize the moduli space $\mathcal{M}_d = E_d(\mathbb{Z})\backslash E_d(\mathbb{R})/K_d$ [44] where K_d is the maximal compact subgroup of $E_d(\mathbb{R})$. It turns out that $\Phi = \exp(-S_\alpha + i\mathcal{C}_\alpha)$ is a harmonic function on $E_d(\mathbb{R})/K_d$ with respect to the $E_d(\mathbb{R})$-left invariant metric [42, 10].

Furthermore, actions of simple combinations of instantons, corresponding to Wick rotated bound states, are also given by harmonic functions. This observation allows us to algebraically relate bound states of euclidean branes to the Lie algebra roots. For example, an M2-brane that is wrapping the diagonal of the $R_3 - R_4$ torus has action $S = 2\pi M_p^3 R_1 R_2 \sqrt{R_3^2 + R_4^2}$, in the absence of θ-angles. In the presence of θ-angles, this action is, in general, modified to

$$S' = \frac{1}{(2\pi)^2} M_p^3 \int_{M2} \sqrt{g} d^3x + i \int_{M2} C\,,$$

where g is the induced metric on the M2-brane and the integrals are performed on the M2-brane worldvolume. Let us assume that the only nonzero θ-angles are $\mathcal{C}_\alpha \equiv C_{123}$ and $\mathcal{C}_\beta \equiv C_{124}$, where we have introduced the two Lie algebra roots α and β with

$$S_\alpha = 2\pi M_p^3 R_1 R_2 R_3\,, \qquad S_\beta = 2\pi M_p^3 R_1 R_2 R_4\,.$$

The harmonic function then reduces to

$$\Phi' \Longrightarrow e^{-S' + i(\mathcal{C}_\alpha + \mathcal{C}_\beta)}\,, \qquad S' = \sqrt{S_\alpha^2 + S_\beta^2}\,.$$

The point is that we can determine S' by calculating the absolute value of a harmonic function whose phase behaves as $\exp\{i\mathcal{C}_\alpha + i\mathcal{C}_\beta\}$. (See [10] for more details.)

3. Combinatorics

As we have reviewed in section 2.3, each positive real root α of E_{10} corresponds to a unique euclidean brane, and the action of the brane, in the absence of fluxes, is given by $2\pi \exp\langle \alpha, \vec{h}\rangle$, where $\vec{h}$ is the vector of logs of radii given by (2.13). The actions are of the form $2\pi \prod_{i=1}^{10} (M_p R_i)^{n_i}$ where n_i are positive integers, except for Kaluza-Klein instantons in which case one n_i is -1. From this action we can read off the dimension of the brane by counting the number of powers for which $n_i = 1$. For example for an M2-brane the action could be $2\pi M_p^3 R_1 R_2 R_3$ and the dimension is 3.

Formally, we can define an action corresponding to imaginary roots of E_{10} in exactly the same manner, $2\pi \exp\langle \alpha, \vec{h}\rangle$. We can then ask similar questions, such as how many n_i's are 1, about the imaginary roots as well. The purpose of this section is to study such "combinatorial" properties of the real as well as the imaginary branes. In this section we will naively interpret the imaginary roots as euclidean branes. However, in section 4, we will propose another interpretation that we believe is better.

3.1 Root properties

We will now work with the Lie algebra E_{10}. As we have seen in section 2.1, our convenient basis for the weight space $\mathbb{R}^{10}$ is such that the root lattice $\Gamma \subset \mathbb{R}^{10}$ is spanned by vectors

$$\alpha = (n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10})\,, \qquad n_i \in \mathbb{Z},\ (i = 1, \ldots, 10)\,, \quad \sum_{i=1}^{10} n_i \equiv 0 \bmod 3\,,$$

and such that the Cartan product is given by

$$\alpha^2 = \sum_{i=1}^{10} n_i^2 - \frac{1}{9}\left(\sum_{i=1}^{10} n_i\right)^2 = \frac{1}{9}\sum_{1\le i<j\le 10}(n_i-n_j)^2 - \frac{1}{9}\sum_{i=1}^{10} n_i^2 .$$

The action of the corresponding *formal* brane $\mathcal{B}_\alpha$ is $2\pi\prod_i (M_p R_i)^{n_i}$ where R_i are here best thought of as abstract variables (formally, the radii of the 10 directions of T^{10}). Strictly speaking, this is the action of an instanton in the absence of θ-angles, i.e. off-diagonal metric terms such as $g_{12},\dots$, and in the absence of 3-form fluxes such as C_{123}, etc.

The inner product of two roots α and α' is given by

$$\langle\alpha,\alpha'\rangle \equiv \sum_{i=1}^{10} n_i n_i' - \frac{1}{9}\left(\sum_{i=1}^{10} n_i\right)\left(\sum_{i=1}^{10} n_i'\right)$$

We can translate the actions to type-IIA by defining

$$m_0 \equiv -n_{10} + \frac{1}{3}\sum_{i=1}^{10} n_i\,, \qquad m_i \equiv n_i\,, \quad i = 1,\dots,9\,, \tag{3.1}$$

The action can then *formally* be written as $2\pi {g_s}^{-m_0}\prod_{i=1}^{9}(M_s l_i)^{m_i}$ where g_s is the string coupling constant, M_s is the string scale, and l_i are the *formal* compactification radii. (These formulas again assume that all θ-angles are zero.) The inner product can then be written as

$$\langle\alpha,\alpha'\rangle = 2m_0 m_0' + \sum_{i=1}^{9} m_i m_i' - \frac{1}{2}m_0\sum_{i=1}^{9} m_i' - \frac{1}{2}m_0'\sum_{i=1}^{9} m_i\,.$$

For future use we need to define

Definition 2. We say that a root α with indices n_i is *thicker* (*thinner*) than a root α' with indices n_i' if $n_i \ge n_i'$ ($n_i \le n_i'$) for $i = 1,\dots,10$.

The roots given by

$$\Theta_{1,\dots,9} \equiv (1,1,1,1,1,1,1,1,1,0)$$

and all its permutations are the thinnest among all the imaginary roots.

Definition 3. we define the *void count* of the root α to be the number of i's for which $n_i = 0$.

Definition 4. we define the *singleton count* of the root α to be the number of i's for which $n_i = 1$.

Definition 5. we define the *doubleton count* of the root α to be the number of i's for which $n_i = 2$.

Claim 2. *The only positive imaginary roots with void count > 0 are permutations of the following:*

$$(0,n,n,n,n,n,n,n,n,n)\,, \qquad n > 0\,.$$

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Root	Square	s	d	multiplicity
$\Theta_{2\ldots 10} \equiv (0,1,1,1,1,1,1,1,1,1)$	$(\Theta_{2\ldots 10})^2 = 0$	9	0	$m = 8$
$\Theta_{1\ldots 8;9\,10} \equiv (1,1,1,1,1,1,1,1,2,2)$	$(\Theta_{1\ldots 8;9\,10})^2 = 0$	8	2	$m = 8$
$\Theta_{1\ldots 5;6\ldots 10} \equiv (1,1,1,1,1,2,2,2,2,2)$	$(\Theta_{1\ldots 5;6\ldots 10})^2 = 0$	5	5	$m = 8$
$\Theta_{123;4\ldots 9;10} \equiv (1,1,1,2,2,2,2,2,2,3)$	$(\Theta_{123;4\ldots 9;10})^2 = 0$	3	6	$m = 8$
$\Theta_{12;3\ldots 7;89\,10} \equiv (1,1,2,2,2,2,2,3,3,3)$	$(\Theta_{12;3\ldots 7;89\,10})^2 = 0$	2	5	$m = 8$
$\Theta_{12;34;5\ldots 10} \equiv (1,1,2,2,3,3,3,3,3,3)$	$(\Theta_{12;34;5\ldots 9\,10})^2 = 0$	2	2	$m = 8$
$\Theta_{12;;3\ldots 9;10} \equiv (1,1,3,3,3,3,3,3,3,4)$	$(\Theta_{12;;3\ldots 9;10})^2 = 0$	2	0	$m = 8$
$\Theta_{12;3\ldots 10} \equiv (1,1,2,2,2,2,2,2,2,2)$	$(\Theta_{12;3\ldots 10})^2 = -2$	2	8	$m = 44$

Table 1: Imaginary roots of E_{10} with singleton count≥ 2.

Proof. Without loss of generality we may assume that $n_1 = 0$. The root is then an element of the $\mathbf{g} = E_9$ subalgebra. The claim immediately follows from the characterization of the imaginary roots of E_9 as $n\delta$. □

Claim 3. *A positive imaginary root has no negative n_i's. A positive real root has negative n_i's only if it is a permutation of $(1,-1,0,\ldots,0)$.*

The proof is given in the appendix.

Let us now describe the imaginary roots α up to the action of the Weyl group.

Proposition 6. *Every positive imaginary root $\gamma \in \hat{\Delta}^+_{\mathrm{Im}}$ of $\hat{\mathbf{g}} = E_{10}$ can be uniquely written as $\gamma = w(\alpha)$ with $w \in \hat{W}$ an element of the Weyl group, and $\alpha \in \hat{Q}$ given by*

$$\alpha = (n_1, n_2, \ldots, n_{10}) \,,$$

and satisfying

$$0 < n_1 \leq n_2 \leq \cdots \leq n_{10} \,, \qquad 2(n_8 + n_9 + n_{10}) \leq n_1 + n_2 + \cdots + n_7 \,.$$

Proof. This follows immediately from proposition 1 and claim 3. □

Theorem 1. *The only imaginary roots with a singleton count $s \geq 2$ are permutations of the roots given in table 1. In that table, we have indicated the square of the root, the singleton count s, the doubleton count d, and the multiplicity of the root m. There is an infinite number of imaginary roots of singleton count $s = 1$.*

Proof. The proofs are given in the appendix.

Our notation $\Theta_{i_1 i_2,\ldots,i_s;j_1 j_2,\ldots,j_d;\ldots}$ indicates the indices $i_1,\ldots,i_s$ that have $n_{i_1} = \cdots = n_{i_s} = 1$, then the indices $j_1, j_2, \ldots, j_d$ that have $n_{j_1} = \cdots = n_{j_d} = 2$, and so on. By analogy with instantonic branes associated to real roots, we will say that *the imaginary root is extended in directions $i_1,\ldots,i_s$*. Ironically, in section 4 we will argue that such a root corresponds to a brane that is *not* extended in these directions. But, for the naive interpretation of the present section the terminology above is natural.

Root	Square	s	d	multiplicity
$\Theta_{89\,10} \equiv (0,0,0,0,0,0,0,1,1,1)$	$(\Theta_{89\,10})^2 = 2$	3	0	$m=1$
$\Theta_{5\ldots10} \equiv (0,0,0,0,1,1,1,1,1,1)$	$(\Theta_{5\ldots10})^2 = 2$	6	0	$m=1$
$\Theta_{3\ldots9;10} \equiv (0,0,1,1,1,1,1,1,1,2)$	$(\Theta_{3\ldots9;10})^2 = 2$	7	1	$m=1$
$\Theta_{2\ldots7;89\,10} \equiv (0,1,1,1,1,1,1,2,2,2)$	$(\Theta_{2\ldots7;89\,10})^2 = 2$	6	3	$m=1$
$\Theta_{234;5\ldots10} \equiv (0,1,1,1,2,2,2,2,2,2)$	$(\Theta_{234;5\ldots10})^2 = 2$	3	6	$m=1$
$\Theta_{1\ldots9;;10} \equiv (1,1,1,1,1,1,1,1,1,3)$	$(\Theta_{1\ldots9;;10})^2 = 2$	9	0	$m=1$
$\Theta_{1\ldots6;789;10} \equiv (1,1,1,1,1,1,2,2,2,3)$	$(\Theta_{1\ldots6;789;10})^2 = 2$	6	3	$m=1$
$\Theta_{1\ldots4;5\ldots8;9\,10} \equiv (1,1,1,1,2,2,2,2,3,3)$	$(\Theta_{1\ldots4;5\ldots8;9\,10})^2 = 2$	4	4	$m=1$
$\Theta_{123;456;7\ldots10} \equiv (1,1,1,2,2,2,3,3,3,3)$	$(\Theta_{123;456;7\ldots10})^2 = 2$	3	3	$m=1$
$\Theta_{123;;4\ldots10} \equiv (1,1,1,3,3,3,3,3,3,3)$	$(\Theta_{123;;4\ldots10})^2 = 2$	3	0	$m=1$
$\Theta_{12;3\ldots8;9;10} \equiv (1,1,2,2,2,2,2,2,3,4)$	$(\Theta_{12;3\ldots8;9;10})^2 = 2$	2	6	$m=1$
$\Theta_{12;345;6\ldots9;10} \equiv (1,1,2,2,2,3,3,3,3,4)$	$(\Theta_{12;345;6\ldots9;10})^2 = 2$	2	3	$m=1$
$\Theta_{12;3;4\ldots8;9\,10} \equiv (1,1,2,3,3,3,3,3,4,4)$	$(\Theta_{12;3;4\ldots8;9\,10})^2 = 2$	2	1	$m=1$
$\Theta_{12;;3\ldots6;7\ldots10} \equiv (1,1,3,3,3,3,4,4,4,4)$	$(\Theta_{12;;3\ldots6;7\ldots10})^2 = 2$	2	0	$m=1$
$\Theta_{12;;3;4\ldots10} \equiv (1,1,3,4,4,4,4,4,4,4)$	$(\Theta_{12;;3;4\ldots10})^2 = 2$	2	0	$m=1$

Table 2: Real roots of E_{10} with singleton count≥ 2.

The formal actions associated with the imaginary roots are

$$\begin{aligned}
S_{2\ldots10} &= 2\pi M_p^9 R_2\cdots R_{10}, \\
S_{1\ldots8;9\,10} &= 2\pi M_p^{12} R_1\cdots R_8(R_9R_{10})^2, \\
S_{1\ldots5;6\ldots10} &= 2\pi M_p^{15} R_1\cdots R_5(R_6\cdots R_{10})^2, \\
S_{123;4\ldots9;10} &= 2\pi M_p^{18} R_1R_2R_3(R_4\cdots R_9)^2R_{10}^3, \\
S_{12;3\ldots7;89\,10} &= 2\pi M_p^{21} R_1R_2(R_3\cdots R_7)^2(R_8R_9R_{10})^3, \\
S_{12;34;5\ldots10} &= 2\pi M_p^{24} R_1R_2(R_3R_4)^2(R_5\cdots R_{10})^3, \\
S_{12;;3\ldots9;10} &= 2\pi M_p^{27} R_1R_2(R_3\cdots R_9)^3R_{10}^4, \\
S_{12;3\ldots10} &= 2\pi M_p^{18} R_1R_2(R_3\cdots R_{10})^2.
\end{aligned} \tag{3.2}$$

For completeness we present:

Theorem 2. *The only real roots with a singleton count $s \geq 2$ are permutations of the roots given in table 2. There is an infinite number of real roots of singleton count $s = 1$.*

The proof is also outlined in the appendix, and see also [7, 46]. Note that $\Theta_{89\,10}$ corresponds to an M2-brane, $\Theta_{5\ldots10}$ to an M5-brane, $\Theta_{3\ldots9;10}$ to a Kaluza-Klein monopole, and $\Theta_{1\ldots9;;10}$ becomes a D8-brane after reduction to type-IIA on the 10^{th} direction, as in equation (3.1).

Definition 6. we define the *hyperplane of the root* to be the subspace of $\mathbb{R}^{10}$ generated by unit vectors in all directions i for which $n_i = 1$.

Obviously, the hyperplane of the root has a dimension equal to the singleton count.

In the notation and terminology of section 2.1.1, the imaginary roots listed above can be constructed as follows. We first write down all the positive *real* roots that can be obtained from the simple root α_{-1} (with corresponding action $2\pi R_1 R_2^{-1}$) by Weyl reflections in the Weyl group W of E_9. These reflections are generated by the simple reflections $r_0, \ldots, r_8$. The simple reflections $r_0, \ldots, r_7$ act simply as permutations of the indices of $R_2, \ldots, R_9$ and we can ignore them. Successive application of the simple reflection r_8 on α_{-1}, with suitable permutations of the indices in between, produces the following list of real roots:

$$
\begin{aligned}
S_{\alpha_{-1}} &= 2\pi R_1 R_2^{-1}, \\
r_8 \cdots r_j(\alpha_{-1}) &= \left(\sum_{i=-1}^{5} \alpha_i\right) + \alpha_8 \Rightarrow S = 2\pi M_p^3 R_1 R_9 R_{10}; \\
r_8 \cdots r_j r_8 \cdots r_k(\alpha_{-1}) &= \left(\sum_{i=-1}^{8} \alpha_i\right) + \alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_8 \\
&\Rightarrow S = 2\pi M_p^6 R_1 R_6 R_7 R_8 R_9 R_{10}; \\
r_8 \cdots r_j r_8 \cdots r_k r_8 \cdots r_l(\alpha_{-1}) &= \alpha_{-1} + 2\alpha_0 + 3\alpha_1 + 4(\alpha_2 + \alpha_3 + \alpha_4) + 5\alpha_5 + 3\alpha_6 + \alpha_7 + 3\alpha_8 \\
&\Rightarrow S = 2\pi M_p^9 R_1 R_2 R_3 R_4 R_7 R_8 R_9 R_{10}^2; \quad (j, k, l = 0, \ldots, 7)\,.
\end{aligned}
$$

We know from section 2.1.2 that all of the remaining level-1 roots are simply translations of the maximal weights of $L(-\alpha_{-1})$ by $-n\delta$ for $n \geq 1$. (Recall that $S_{n\delta} = 2\pi(M_p^9 R_2 R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10})^n$.)

For $n = 1$ these translations give us the following imaginary roots with multiplicity $m = 8$:

$$\Theta_{13456789\,10}\,(s = 9), \quad \Theta_{12345678;9\,10}\,(s = 8), \quad \Theta_{12345;6789\,10}\,(s = 5), \quad \Theta_{156;234789;10}\,(s = 3),$$

For $n = 2$ these translations give us the following imaginary roots with multiplicity $m = 44$:

$$\Theta_{12;3456789\,10}(s = 2), \quad \Theta_{1;2345678;9\,10}(s = 1), \quad \Theta_{1;2345;6789\,10}(s = 1), \quad \Theta_{1;56;234789;10}(s = 1),$$

The last three roots did not appear in our table 1 above since their singleton count is smaller than 2.

Translating the root actions above to type-IIA notation we obtain

$$
S_{1\ldots9} \rightarrow \begin{cases} \frac{2\pi}{g_s{}^2} M_s{}^8 l_1 \cdots l_8 \\ \frac{2\pi}{g_s{}^3} M_s{}^9 l_1 \cdots l_9 \end{cases}
\qquad
S_{1\ldots8;9\,10} \rightarrow \begin{cases} \frac{2\pi}{g_s{}^2} M_s{}^{10} l_1 \cdots l_8 l_9^2 \\ \frac{2\pi}{g_s{}^3} M_s{}^{11} l_1 \cdots l_7 (l_8 l_9)^2 \end{cases}
$$

$$
S_{1\ldots5;6789\,10} \rightarrow \begin{cases} \frac{2\pi}{g_s{}^3} M_s{}^{13} l_1 \cdots l_5 (l_6 \cdots l_9)^2 \\ \frac{2\pi}{g_s{}^4} M_s{}^{14} l_1 \cdots l_4 (l_5 \cdots l_9)^2 \end{cases}
\qquad
S_{123;456789;10} \rightarrow \begin{cases} \frac{2\pi}{g_s{}^3} M_s{}^{15} l_1 \cdots l_3 (l_4 \cdots l_9)^2 \\ \frac{2\pi}{g_s{}^4} M_s{}^{16} l_1 \cdots l_3 (l_4 \cdots l_8)^2 l_9^3 \\ \frac{2\pi}{g_s{}^5} M_s{}^{17} l_1 l_2 (l_3 \cdots l_8)^2 l_9^3 \end{cases}
$$

$$
S_{12;3456789\,10} \rightarrow \begin{cases} \frac{2\pi}{g_s{}^4} M_s{}^{16} l_1 l_2 (l_3 \cdots l_9)^2 \\ \frac{2\pi}{g_s{}^5} M_s{}^{17} l_1 (l_2 \cdots l_9)^2 \end{cases}
\qquad
S_{12;34567;89\,10} \rightarrow \begin{cases} \frac{2\pi}{g_s{}^4} M_s{}^{18} l_1 l_2 (l_3 \cdots l_7)^2 (l_8 l_9)^3 \\ \frac{2\pi}{g_s{}^5} M_s{}^{19} l_1 l_2 (l_3 \cdots l_6)^2 (l_7 l_8 l_9)^3 \\ \frac{2\pi}{g_s{}^6} M_s{}^{20} l_1 (l_2 \cdots l_6)^2 (l_7 l_8 l_9)^3 \end{cases}
$$

$$
S_{12;34;56789\,10} \rightarrow \begin{cases} \frac{2\pi}{g_s{}^5} M_s{}^{21} l_1 l_2 (l_3 l_4)^2 (l_5 \cdots l_9)^3 \\ \frac{2\pi}{g_s{}^6} M_s{}^{22} l_1 l_2 l_3^2 (l_4 \cdots l_9)^3 \\ \frac{2\pi}{g_s{}^7} M_s{}^{23} l_1 (l_2 l_3)^2 (l_4 \cdots l_9)^3 \end{cases}
$$

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3.2 Orthogonal roots

For two real roots α, β, the condition $(\alpha|\beta) = 0$ has a physical interpretation in terms of the corresponding instantons [57]–[61, 10, 12]. It means that the two instantons can "bind at threshold," [62, 63] so that the bound instanton has only one time-translation zero-mode and its action is the sum of the actions of the two individual instantons (at least when all the θ-angles are set to zero). For example, an M2-brane with action $2\pi M_p^3 R_1 R_4 R_5$ can bind at threshold to an M2-brane with action $2\pi M_p^3 R_1 R_2 R_3$. It can also bind at threshold to a Kaluza-Klein instanton with action $2\pi R_2 R_4^{-1}$, and so on. We will now calculate which imaginary roots from the lists above are orthogonal to various real roots.

An imaginary root that is extended in directions $1, \ldots, s$ (see the definitions above) is orthogonal to all the real roots corresponding to Kaluza-Klein instantons that have actions $2\pi R_k R_l^{-1}$ for $1 \leq k < l \leq s$.

As another example, let α be a real root that corresponds to an M2-brane instanton. It is orthogonal to $\Theta_{1\ldots9}$ if the M2-brane's hyperplane is a subset of the $\Theta_{1\ldots9}$'s hyperplane.

The real root α is orthogonal to $\Theta_{1\ldots8;9\,10}$ if their hyperplanes intersect on a dimension-2 plane. In this case, the intersection has co-dimension 1 inside the M2-brane's hyperplane. It is therefore tempting to say that the M2-brane *can end* on the $\Theta_{1\ldots8;9\,10}$-instanton, just like an M2-brane can end on an M5-brane [64, 57].

The real root α is orthogonal to $\Theta_{1\ldots5;6789\,10}$ if their hyperplanes intersect on a dimension-1 hyperplane (a line).

There are two distinct possibilities for α to be orthogonal to $\Theta_{123;456789;10}$. In one, the corresponding hyperplanes intersect along a line, and in the other the hyperplanes intersect only at the origin.

Similarly, an M2-brane root α is orthogonal to $\Theta_{12;34567;89\,10}$ in two distinct cases. In one, the intersection of their hyperplanes is exactly the origin, and in the other the intersection is a dimension-1 hyperplane (a line).

An M2-brane root α is orthogonal to $\Theta_{12;34;56789\,10}$ or $\Theta_{12;3456789\,10}$ only if the intersection of their hyperplanes is exactly the origin.

We can perform a similar analysis for a root α that corresponds to an M5-brane, but we will not present it here.

4. Physical interpretation of imaginary roots

The discussion in section 3 assumed that imaginary roots correspond to euclidean branes. We can always *define* the action corresponding to an imaginary root as $2\pi \exp\langle\alpha, \vec{h}\rangle$, as we did in section 3, and study its combinatorial properties. But this definition lacks sufficient physical motivation.

In this section we would like to propose an alternative interpretation that, we believe, is more physical. We propose that a prime positive imaginary root γ with $\gamma^2 = 0$ corresponds to a Minkowski brane, and $2\pi \exp\langle\vec{h}, \gamma\rangle$ describes its mass in units inverse to conformal time. We will begin to study the imaginary roots γ by looking for two real positive roots α and β such that $\gamma = \alpha + \beta$.

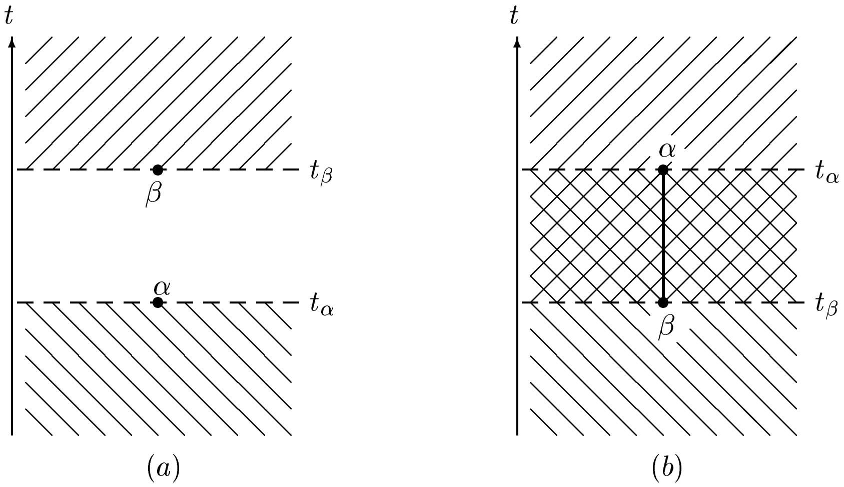

Figure 1: Two instantons associated with the real roots α, β. Each instanton creates a jump in the associated flux. The fluxes are depicted by the diagonal pattern. Instanton α creates a jump from a nonzero value to 0, while instanton β changes another flux from 0 to a nonzero value. (a) Instanton α occurs before instanton β, and the different fluxes do not overlap; (b) Instanton α occurs after instanton β, and the fluxes overlap between times t_β and t_α. In addition, a particle (the thick vertical line) associated to $\gamma = \alpha + \beta$ is created between the two instantons.

4.1 Prime roots with $\gamma^2 = 0$

Let $\Theta_{2\ldots10}$ be the imaginary root that corresponds to the action

$$S_{\Theta_{2\ldots10}} = 2\pi e^{\langle \delta, \vec{h} \rangle} = 2\pi M_p^9 R_2 R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10} \,.$$

It satisfies $(\Theta_{2\ldots10})^2 = 0$, and it is minimal in the sense that no other imaginary root is thinner (see section 3).

We will start with the roots that can be obtained from $\Theta_{2\ldots10}$ by a Weyl reflection. These are all the prime roots that square to zero. For a specific example, take α, β corresponding to M5-branes with actions

$$S_\alpha = 2\pi e^{\langle \alpha, \vec{h} \rangle} = 2\pi M_p^6 R_1 R_2 R_3 R_4 R_5 R_6 \,, \qquad S_\beta = 2\pi e^{\langle \beta, \vec{h} \rangle} = 2\pi M_p^6 R_1 R_2 R_7 R_8 R_9 R_{10} \,, \tag{4.1}$$

Then $\gamma = \alpha + \beta$ is an imaginary root with $\gamma^2 = 0$ and multiplicity $m = 8$ and

$$S_\gamma = 2\pi M_p^{12} (R_1 R_2)^2 R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10} \,. \tag{4.2}$$

Let $J^{+\alpha}$ and $J^{+\beta}$ be elements of E_{10} that correspond to the real roots. Then the commutator $[J^{+\alpha}, J^{+\beta}]$ is in the weight space corresponding to γ.

Physically, α and β correspond to M5-brane instantons. Let α be an instanton at time t_α and β at time t_β. Now consider switching the time order of the two instantons from, say, $t_\alpha \ll t_\beta$ to $t_\alpha \gg t_\beta$ (see figure 1). In this "process" one M5-brane passes through the other. But this is precisely the M2-brane creation process described in [48]. After the process there is an extra M2-brane stretched along the $1^{st}, 2^{nd}$ directions and extended in time from t_β to t_α.

The brane creation process has various versions for different roots. We will now describe a few of the versions. In the setting that we described above, the creation of the M2-branes can be argued as follows. The instanton at t_β creates a jump in the flux $G_{789\,10}$ and the instanton at t_α creates a jump in the flux G_{3456} so that

$$(2\pi)^3 G_{789\,10} = \begin{cases} N & \text{for } t < t_\beta, \\ N+1 & \text{for } t > t_\beta, \end{cases} \qquad (2\pi)^3 G_{3456} = \begin{cases} N'-1 & \text{for } t < t_\alpha, \\ N' & \text{for } t > t_\alpha, \end{cases}$$

for some integers N, N'. As we recalled in section 2.2, the $\int C \wedge G \wedge G$ term of 10+1D supergravity indicates that $G \wedge G$ is a source for M2-brane flux and there must be an equal number of anti- M2-branes to cancel that flux [52]. Therefore, together with the instanton at t_β, $N'-1$ anti- M2-branes must also be present if $t_\beta < t_\alpha$ and N anti- M2-branes must be present if $t_\beta > t_\alpha$. Setting $N = N' = 0$ we see that one M2-brane is stretched between the two instantons if $t_\beta < t_\alpha$.

There is a U-dual process involving geometry alone [65, 66]. In this case we take β to correspond to an M2-brane and α to correspond to a Kaluza-Klein monopole such that

$$S_{\beta'} = 2\pi e^{\langle \beta', \vec{h} \rangle} = 2\pi M_p^3 R_2 R_9 R_{10}\,, \qquad S_{\alpha'} = 2\pi e^{\langle \alpha', \vec{h} \rangle} = 2\pi M_p^9 R_1^2 R_2 R_3 R_4 R_5 R_6 R_7 R_8, \tag{4.3}$$

Then $\gamma = \alpha' + \beta'$ the same as before. This time the process of M2-brane creation can be understood entirely from the geometry of the Kaluza-Klein monopole. The Kaluza-Klein monopole changes by one unit the first Chern class c_1 of the fibration of the 1^{st} circle over the T^2 in the 9^{th} and 10^{th} directions. Suppose that

$$c_1 = \begin{cases} 0 & \text{for } t < t_\alpha. \\ 1 & \text{for } t > t_\alpha. \end{cases}$$

Then the M2-brane in the $2^{nd}, 9^{th}, 10^{th}$ directions cannot pass through the Kaluza-Klein monopole. It must get "stuck" at some point along the $9^{th} - 10^{th}$ plane, and it is not hard to see that an M2-brane that wraps the 1^{st} and 2^{nd} directions is created.

Another U-dual process involves passing a D0-brane through a D8-brane [67] or a D4-brane through another D4-brane [68, 69]. In these processes a string is created. To relate it to our E_{10} conventions, we lift type-IIA to M-theory, taking momentum in the 2^{nd} direction to be related to D0-brane charge. We then take the roots α'' and β'' as follows:

$$S_{\alpha''} = 2\pi e^{\langle \alpha'', \vec{h} \rangle} = 2\pi R_1 R_2^{-1}\,, \qquad S_{\beta''} = 2\pi e^{\langle \beta'', \vec{h} \rangle} = 2\pi M_p^{12} R_1 R_2^3 R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10}\,. \tag{4.4}$$

Again, $\gamma = \alpha'' + \beta''$ is the same as before and also the object that is created is the same M2-brane stretched in the 1^{st} and 2^{nd} directions.

We conclude that the root γ with

$$S_\gamma = 2\pi e^{\langle \gamma, \vec{h} \rangle} = 2\pi M_p^{12} (R_1 R_2)^2 R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10}$$

corresponds to a physical (temporally extended) M2-brane stretched in the 1^{st} and 2^{nd} directions.

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For another example, take γ with

$$S_\gamma = 2\pi e^{\langle\gamma,\vec{h}\rangle} = 2\pi M_p^9 R_2 R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10}\,.$$

We can decompose it as a sum of two real roots as $\gamma = \alpha + \beta$ with

$$S_\alpha = 2\pi M_p^3 R_2 R_3 R_4\,, \qquad S_\beta = 2\pi M_p^6 R_5 R_6 R_7 R_8 R_9 R_{10}\,.$$

An instanton corresponding to α creates a jump by one unit in the flux G_{0234} and an instanton corresponding to β creates a jump by one unit in the flux G_{1234}. When the two fluxes G_{0234} and G_{1234} are present together, we get a contribution to the field-theoretic momentum $P^1 \equiv \int \sqrt{g} G^{0\mu_1\mu_2\mu_3} G^1{}_{\mu_1\mu_2\mu_3} d^{10}x$. Since the total momentum must be zero, there must be extra Kaluza-Klein particles with the opposite amount of momentum. Thus, γ corresponds to a Kaluza-Klein particle with momentum in the 1^{st} direction. In section 4.2 we will write down a mass formula for the physical objects corresponding to the imaginary roots γ that will allow us to immediately see that γ above corresponds to a Kaluza-Klein particle with mass R_1^{-1}.

4.2 A mass formula

There is a simple formula that relates the imaginary root γ to the action of the physical brane. Let us list the branes that we found and their "masses," i.e. actions per unit time dt. Let us write down the first four roots from table 1 or equation (3.2), together with the masses of their corresponding branes (that we denote by M'),

$$\begin{aligned}
e^{\langle\Theta_{2\ldots10},\vec{h}\rangle} &= M_p^9 R_2 R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10}, && M' = R_1^{-1},\\
e^{\langle\Theta_{1\ldots8;9\,10},\vec{h}\rangle} &= M_p^{12} R_1 R_2 R_3 R_4 R_5 R_6 R_7 R_8 (R_9 R_{10})^2, && M' = M_p^3 R_9 R_{10},\\
e^{\langle\Theta_{12345;6789\,10},\vec{h}\rangle} &= M_p^{15} R_1 R_2 R_3 R_4 R_5 (R_6 R_7 R_8 R_9 R_{10})^2, && M' = M_p^6 R_6 R_7 R_8 R_9 R_{10},\\
e^{\langle\Theta_{123;456789;10},\vec{h}\rangle} &= M_p^{18} R_1 R_2 R_3 (R_4 R_5 R_6 R_7 R_8 R_9)^2 R_{10}^3, && M' = M_p^9 R_4 R_5 R_6 R_7 R_8 R_9 R_{10}^2.
\end{aligned} \tag{4.5}$$

The roots in equation (4.5) correspond to a Kaluza-Klein particle, M2-brane, M5-brane, and Kaluza-Klein monopole, respectively. The mass M' can be written as

$$M' = \frac{e^{\langle\gamma,\vec{h}\rangle}}{M_p^9 V_{10}}\,, \qquad V_{10} \equiv R_1 \cdots R_{10}.$$

The factor V_{10} might seem strange at first, but if we recall the definition of conformal time (2.14), we can write the Minkowski action $\tilde{S}_\gamma$ of the brane per unit conformal time as

$$\frac{d\tilde{S}_\gamma}{d\tilde{\tau}} = 2\pi M_p^9 V_{10} \frac{dS_\gamma}{dt} = 2\pi M_p^9 V_{10} M' = 2\pi e^{\langle\gamma,\vec{h}\rangle}\,. \tag{4.6}$$

We will refer to this equation as the *mass formula*.

The remaining roots from table 1 or equation (3.2) are

$$\begin{aligned}
e^{\langle\Theta_{12;34567;89\,10},\vec{h}\rangle} &= M_p^{21} R_1 R_2 (R_3\cdots R_7)^2 (R_8 R_9 R_{10})^3, && M' = M_p^{12} R_3 \cdots R_7 (R_8 R_9 R_{10})^2,\\
e^{\langle\Theta_{12;34;56789\,10},\vec{h}\rangle} &= M_p^{24} R_1 R_2 (R_3 R_4)^2 (R_5\cdots R_{10})^3, && M' = M_p^{15} R_3 R_4 (R_5 \cdots R_{10})^2,
\end{aligned}$$

$$e^{\langle\Theta_{12;;3456789;10},\vec{h}\rangle} = M_p^{27}R_1R_2(R_3\cdots R_9)^3R_{10}^4, \qquad M' = M_p^{18}(R_3\cdots R_9)^2R_{10}^3,$$
$$e^{\langle\Theta_{12;3456789\,10},\vec{h}\rangle} = M_p^{18}R_1R_2(R_3R_4R_5R_6R_7R_8R_9R_{10})^2, \quad M' = M_p^{9}R_3R_4R_5R_6R_7R_8R_9R_{10}. \tag{4.7}$$

The roots in equation (4.7) are unfamiliar objects, but the first three roots are Weyl reflections (formally U-duals) of the roots of (4.5). Note that the expressions for the masses of $\Theta_{12;34567;89\,10}, \Theta_{12;34;56789\,10}$ can be obtained from the actions of the real roots $\Theta_{2\ldots7;89\,10}, \Theta_{234;5\ldots10}$ (see table 2) as follows

$$M'\left\{\Theta_{12;34567;89\,10}\right\} = \frac{S\left\{\Theta_{2\ldots7;89\,10}\right\}}{2\pi R_2}, \qquad M'\left\{\Theta_{12;34;56789\,10}\right\} = \frac{S\left\{\Theta_{234;5\ldots10}\right\}}{2\pi R_2}.$$

This is in agreement with our physical interpretation of the real roots as instantons. The imaginary roots can be obtained by Wick rotating an instanton back to Minkowski space. If we replace R_2 with the time direction we can formally convert the instantons to the Minkowski branes associated with the two imaginary roots $\Theta_{12;34567;89\,10}, \Theta_{12;34;56789\,10}$.

The third imaginary root $\Theta_{12;;3456789;10}$ in (4.7) can be obtained in a similar way from the real root $\Theta_{2;3456789;10}$. The latter does not appear in table 2) because its singleton count is $s = 1$, but it can be written as $\delta + \Theta_{3\ldots9;10}$, and $\Theta_{3\ldots9;10}$ appears in table 2 as the root corresponding to a Kaluza-Klein monopole.

The last root satisfies $(\Theta_{12;3456789\,10})^2 = -2$ and so cannot be a Weyl reflection of the other roots (that square to zero). It can be obtained by a Wick rotation similar to the one discussed above, but we have to start with $\delta \equiv \Theta_{2\ldots10}$ which is an imaginary rather than a real root, and therefore does not correspond to an instanton. The physical interpretation of $\Theta_{12;3456789\,10}$ is therefore different. We will return to it in section 4.4.

4.3 The multiplicity

The imaginary roots that we studied in section 4.1 have a multiplicity of $m = 8$. This means that the Lie algebra E_{10} has 8 different generators for the same root. The root determines the commutation relations of these generators with the Cartan subalgebra $\hat{\mathfrak{h}}_{\mathbb{R}}$, and determines the mass of the brane (4.6). Thus, all $m = 8$ generators with the same root yield the same mass. In fact, from the brane creation process discussed in section 4.1 it is obvious that all $m = 8$ generators correspond to the same object.

For example, we constructed an M2-brane stretched in the 1^{st} and 2^{nd} directions with γ given by (4.2), using the two instantons α, β given by (4.1). The root γ was imaginary with multiplicity $m = 8$ and satisfied $\gamma = \alpha + \beta$. The natural Lie algebra generator to associate with this root is (up to a multiplicative factor) the commutator $[J^{+\alpha}, J^{+\beta}]$, where $J^{+\alpha}$ and $J^{+\beta}$ are the generators associated with the roots α, β. They are unique since α, β are real roots with multiplicity $m = 1$. But in (4.3) we decomposed $\gamma = \alpha' + \beta'$ as a sum of different real roots. It is not hard to check that $[J^{+\alpha'}, J^{+\beta'}]$ is linearly independent of $[J^{+\alpha}, J^{+\beta}]$. (For this purpose, note that a Weyl transformation in the Weyl group $\hat{W}$ can be found that simultaneously maps all the roots $\alpha, \beta, \alpha', \beta', \gamma$ to roots inside $\mathbf{g} = E_9$, which is tractable.) Similarly, in (4.4) we constructed yet a third decomposition $\gamma = \alpha'' + \beta''$ which (as is easy to check) yields another linearly independent generator.

Thus, it seems that it is the root that corresponds to the brane and not the generator. In the following subsection we will see that the situation is probably different for roots with negative norm.

4.4 Roots with $\gamma^2 < 0$

Take $\gamma = \alpha + \beta$ with

$$S_\alpha = 2\pi e^{\langle \alpha, \vec{h} \rangle} = 2\pi M_p^9 R_2 R_4^2 R_5 R_6 R_7 R_8 R_9 R_{10},$$
$$S_\beta = 2\pi e^{\langle \beta, \vec{h} \rangle} = 2\pi M_p^9 R_1 R_3^2 R_5 R_6 R_7 R_8 R_9 R_{10}.$$

Then $\gamma^2 = -2$, and

$$S_\gamma = 2\pi e^{\langle \gamma, \vec{h} \rangle} = 2\pi M_p^{18} R_1 R_2 (R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10})^2.$$

This is the root $\gamma = \Theta_{12;3\dots10}$ that puzzled us at the end of section 4.2.

We need to understand what happens when instanton α is pushed past instanton β. Instanton α creates a jump in the first Chern class c_1 of the fibration of the 4^{th} circle over the 1^{st} and 3^{rd} directions while β creates a jump in the first Chern class of the fibration of the 3^{rd} circle over the 2^{nd} and 4^{th} directions.

We are mainly interested in the topology of the manifold. Let (x_1, x_2, x_3, x_4) be the relevant periodic coordinates with $0 \le x_1, \dots, x_4 < 2\pi$. We will describe the manifold as a T^2 fibration over T^2 with the base B spanned by x_1, x_2 and the fiber F spanned by x_3, x_4. We denote a generic point of the fiber by $p \equiv (x_3, x_4)$. A point on $T^4 = B \times F$ is denoted by (x_1, x_2, p).

Let us first discuss the effect of a single instanton, say α. Pick an arbitrary coordinate $0 < a < 2\pi$. The geometry associated with α can be described by cutting the base B along the circle $x_1 = a$ and gluing the part at $x_1 = a - \epsilon$ (for some small $\epsilon > 0$) to the part at $x_1 = a + \epsilon$ by

$$(a - \epsilon, x_2, p) \mapsto (a + \epsilon, x_2, M_\alpha(p)), \qquad 0 \le x_2 < 2\pi, \quad p \in F, \quad M_\alpha \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Here $M_\beta \in \mathrm{SL}(2, \mathbb{Z})$ is a linear transformation acting on the T^2 fiber.

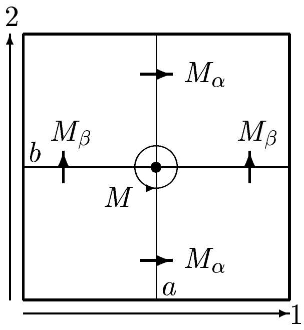
Similarly, the effect of instanton β is described by picking an arbitrary $0 \le b \le 2\pi$, cutting the base B along $x_2 = b$ and gluing according to

$$(x_1, b - \epsilon, p) \mapsto (x_1, b + \epsilon, M_\beta(p)), \qquad 0 \le x_1 < 2\pi, \quad p \in F, \quad M_\beta \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The resulting manifold is smooth except at points that project to $(x_1, x_2) = (a, b)$ on the base. If we go in a circle around (a, b) we discover that the fiber F undergoes a monodromy (see figure 2)

$$M \equiv M_\beta M_\alpha M_\beta^{-1} M_\alpha^{-1} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{4.8}$$

So, we found out that by pushing instanton β past instanton α we create a singularity at $(x_1, x_2) = (a, b)$ that extends in directions $5 \ldots 10$ and is described by a monodromy (4.8) in $\mathrm{SL}(2,\mathbb{Z})$ for the torus in the $3^{rd}, 4^{th}$ directions. This is the same type of monodromies of stringy cosmic strings [70] and F-theory [71]. In fact, setting $M = M_\beta \widetilde{M}$ with $\widetilde{M} \equiv M_\alpha M_\beta^{-1} M_\alpha^{-1}$ we see that, after reducing on the fiber F to type-IIB in the spirit of F-theory, the singularity is that of a $(0,1)$ D7-brane (associated with M_β) and an anti- $(1,1)$ D7-brane (associated with $\widetilde{M}$).



$$M_\alpha \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$M_\beta \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Figure 2: The monodromies in the T^2 fiber, as we pass through cuts on the T^2 base. M is the resulting monodromy around the singular point (a, b).

Note that a different decomposition of $\gamma = \alpha' + \beta'$ with, say,

$$S_{\alpha'} = 2\pi e^{\langle \alpha', \vec{h} \rangle} = 2\pi M_p^9 R_1 R_3 R_4 R_5^2 R_7 R_8 R_9 R_{10} \,,$$
$$S_{\beta'} = 2\pi e^{\langle \beta', \vec{h} \rangle} = 2\pi M_p^9 R_2 R_3 R_4 R_6^2 R_7 R_8 R_9 R_{10} \,, \tag{4.9}$$

yields an apparently different singularity. However, the two decompositions $\gamma = \alpha + \beta$ and $\gamma = \alpha' + \beta'$ define two *different* 1-dimensional subspaces of the 44-dimensional space $\hat{\mathbf{g}}_\gamma$ as follows. If we denote by $J^{+\alpha}, J^{+\beta}, J^{+\alpha'}, J^{+\beta'} \in \hat{\mathbf{g}}$ nonzero Lie algebra elements in $\hat{\mathbf{g}}_\alpha, \ldots, \hat{\mathbf{g}}_{\beta'}$ (unique up to a multiplicative constant) then $[J^{+\alpha}, J^{+\beta}] \in \hat{\mathbf{g}}_\gamma$ and $[J^{+\alpha'}, J^{+\beta'}] \in \hat{\mathbf{g}}_\gamma$ are linearly independent. Thus, in this case it would appear that several different objects are associated with the same root γ, but it might be possible to associate them with different Lie algebra elements in the same space $\hat{\mathbf{g}}_\gamma$.

In any case, the conclusion is that the imaginary root γ is associated with a *pair* of branes of different types (but perhaps not uniquely). It would be interesting to study whether more complicated imaginary roots can be associated with more complicated collections of branes. It is also interesting to note that the affine Lie algebra E_9 and the Kac-Moody E_{10} appeared in the context of configurations of (p, q) 7-branes in the past [34, 72, 35] (and see also [73]).

4.5 Nonprime roots with $\gamma^2 = 0$

According to proposition 2 all imaginary roots with $\gamma^2 = 0$ (called *isotropic*) are $\hat{W}$-equivalent (U-dual) to a multiple of $\delta \equiv \Theta_{2\ldots10}$. We will now discuss the roots $\gamma = n\delta$ with $n > 1$.

Take the case $n = 2$ and decompose $\gamma = \alpha + \beta$ with

$$S_\alpha = 2\pi e^{\langle \alpha, \vec{h} \rangle} = 2\pi M_p^9 R_2^2 R_4 R_5 R_6 R_7 R_8 R_9 R_{10} \,,$$
$$S_\beta = 2\pi e^{\langle \beta, \vec{h} \rangle} = 2\pi M_p^9 R_3^2 R_4 R_5 R_6 R_7 R_8 R_9 R_{10} \,,$$

Then $\gamma = 2\delta$, and

$$S_\gamma = 2\pi e^{\langle \gamma, \vec{h} \rangle} = 2\pi M_p^{18} (R_2 R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10})^2 .$$

In this case, an interpretation of γ via a brane creation process does not work. If we try to mimic the discussion of section 4.4, we discover that the two instantons can pass through each other unharmed. Indeed, this time instanton α creates a jump in the first Chern class c_1 of the fibration of the 2^{nd} circle over the 1^{st} and 3^{rd} directions while β creates a jump in the first Chern class of the fibration of the 3^{rd} circle over the 1^{st} and 2^{nd} directions.

We can create a nontrivial circle fibration of the 3^{rd} direction over the $1-2$ plane by cutting a small disc around the origin of the $1-2$ plane, say of radius $\epsilon > 0$, and gluing it back with a twist

$$(x_1 = \epsilon \cos\theta, x_2 = \epsilon \sin\theta, x_3) \mapsto (\epsilon\cos\theta, \epsilon\sin\theta, x_3 + \theta).$$

Now let us put the two instantons together. Start with $T^3 = S^1 \times S^1 \times S^1$ with directions $1, \dots, 3$. We can simulate the effect of α as follows. Define

$$\Sigma_\alpha \stackrel{\text{def}}{=} \{(0, x_2, 0) : \ 0 \le x_2 < 2\pi\} \subset T^3 .$$

Let $\mathcal{N}_\alpha$ be a small tubular neighborhood of Σ_α. Its topology is $\mathbf{D} \times S^1$ where $\mathbf{D}$ is the 2-dimensional disc. The boundary of $\mathcal{N}_\alpha$ has topology T^2. We can pick $\mathcal{N}_\alpha$ such that

$$\partial\mathcal{N}_\alpha = \{(\epsilon\cos\theta, x_2, \epsilon\sin\theta) : \ 0 \le x_2 < 2\pi, \ 0 \le \theta < 2\pi\} ,$$

for some small $\epsilon > 0$. Topologically, the effect of α is to cut out $\mathcal{N}_\alpha$ off T^3 and glue it back after a Dehn twist:

$$(\epsilon\cos\theta, x_2, \epsilon\sin\theta) \mapsto (\epsilon\cos\theta, x_2 + \theta, \epsilon\sin\theta).$$

Similarly, β can be simulated by cutting a small tubular neighborhood $\mathcal{N}_\beta$ around

$$\Sigma_\alpha \stackrel{\text{def}}{=} \{(0, 0, x_3) : \ 0 \le x_3 < 2\pi\} \subset T^3 ,$$

and gluing it back with a Dehn twist.

But Σ_α and Σ_β are 1-dimensional. We can therefore deform them so that they do not intersect inside T^3. The two instantons can therefore pass through without affecting each other. (If we had tried the same construction in section 4.4 we would have discovered that Σ_α and Σ_β are 2-dimensional and generically intersect at a point inside T^4.)

The nonprime roots must therefore have another interpretation. We do not know what it is.

4.6 Decomposition of level-1 imaginary roots

We will now show that any level-1 imaginary root can be constructed in the manner above, by interchanging the time order between two instantons.

Claim 4. *Any imaginary root $\gamma \in \hat{\Delta}_{[1]}$ is $\hat{W}$-dual to a sum of two positive real roots.*

Proof. We have to show that there exists $w \in \hat{W}$ (the Weyl group of E_{10}) such that $w(\gamma) = \alpha+\beta$ for $\alpha, \beta \in \hat{\Delta}^+_{\mathrm{Re}}$. Recall from section 2.1.2 that any $\gamma \in \hat{\Delta}_{[1]}$ can be reflected into $\hat{\Delta}_{[1]} \cap P_+$ using the E_9 Weyl group W. Recall that P_+ is the set of positive dominant weights of $E_9 \subset E_{10}$ and can be explicitly written as $P_+ \cap \hat{\Delta}_{[1]} = \{\alpha_{-1} + n\delta : n \geq 0\}$. The latter roots are imaginary for $n \geq 1$. Then, for $n \geq 1$, we can decompose $r_0(\alpha_{-1} + n\delta) = \alpha + \beta$ with

$$\alpha = \alpha_{-1} + \alpha_0 + \alpha_1 \,, \qquad \beta = n\delta - \alpha_1 \,.$$

□

4.7 Decomposition of arbitrary imaginary roots

Now let us discuss the decomposition of more general imaginary roots. Let $\gamma \in \hat{\Delta}^+_{\mathrm{Im}}$ be an imaginary root of E_{10}. Can we decompose it as a sum of two positive real roots, $\gamma = \alpha+\beta$?

The problem of finding α that satisfies

$$\alpha^2 = 2 \,, \qquad (\gamma - \alpha)^2 = 2 \Longrightarrow \gamma^2 = 2(\gamma|\alpha),$$

reduces to an inhomogeneous quadratic Diophantine equation in 9 integer unknowns. (We can, for example, eliminate n_{10} from the linear equation $\gamma^2 = 2(\gamma|\alpha)$ and substitute it in $\alpha^2 = 2$.) For $\gamma^2 < 0$, it is not hard to see that the quadratic form is elliptic. (Over $\mathbb{R}$, the metric on $\hat{\mathfrak{h}}_{\mathbb{R}}$ is equivalent to the lorentzian metric on $\mathbb{R}^{9,1}$. The vector γ is timelike, and therefore the equations $\alpha^2 = 2$ and $\gamma^2 = 2(\gamma|\alpha)$ define an 8-dimensional ellipsoid.) We do not know the general answer, but we have tried to decompose several imaginary roots by computer, and were successful each time.

For physical purposes, it is enough to address the weaker question of whether we can find a Weyl-group element $w \in \hat{W}$ such that $w(\gamma) = \alpha + \beta$ for some $\alpha, \beta \in \hat{\Delta}^+_{\mathrm{Re}}$. We can actually drop the restriction of positivity for α, β, because of the following

Claim 5. *Existence of a decomposition $w(\gamma) = \alpha - \beta$ with $\alpha, \beta \in \hat{\Delta}^+_{\mathrm{Re}}$, implies existence of a decomposition $w(\gamma) = \alpha' + \beta'$ with $\alpha', \beta' \in \hat{\Delta}^+_{\mathrm{Re}}$.*

Proof. Suppose $\gamma = \alpha - \beta$. First assume that the minimal number of simple reflections needed to bring β to a simple root is smaller than or equal to the number required for α. Then, applying this minimal list of simple reflections, we obtain $w'(\gamma) = \alpha'' - \alpha_i$, for some $\alpha'' \in \hat{\Delta}^+_{\mathrm{Re}}$, and some simple root $\alpha_i = r_{\alpha_{i_s}} \circ \cdots \circ r_{\alpha_{i_1}}(\beta)$. To see that $\alpha'' \in \hat{\Delta}^+_{\mathrm{Re}}$, we have to use lemma 3.7 of [41] which states that the only way for a sequence of simple reflections $r_{\alpha_{i_s}} \circ r_{\alpha_{i_{s-1}}} \circ \cdots \circ r_{\alpha_{i_1}}$ to take a positive root to a negative root is by passing through a simple root at some stage $\alpha_j = r_{\alpha_{i_t}} \circ \cdots \circ r_{\alpha_{i_1}}(\alpha)$ $(1 \leq t < s)$. But we assumed that at least s simple reflections are required to turn α into a simple root, so $\alpha'' \in \hat{\Delta}^+_{\mathrm{Re}}$. We can now write $r_{\alpha_i} w'(\gamma) = \alpha_i + r_{\alpha_i}(\alpha'')$. It is easy to see that $\alpha' \equiv r_{\alpha_i}(\alpha'')$ cannot be a simple root (otherwise $\alpha' + \alpha_i$ would be a real root of E_{10}) and therefore, by the same arguments as above, it cannot be a negative root and so must be positive. Setting $\beta' \equiv \alpha_i$ and $w \equiv r_{\alpha_i} \circ w'$ we obtain the requisite decomposition $w(\gamma) = \alpha' + \beta'$.

If it is α that requires the smaller number of simple reflections to turn it into a simple root then, following the same steps as above, we get a decomposition $w(\gamma) = -\alpha' - \beta'$. But,

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according to proposition 5.2 of [41], a positive imaginary root γ cannot be $\hat{W}$-equivalent to a negative imaginary root $-\alpha' - \beta'$ (in sharp contrast to real roots!). So this case is ruled out. $\square$

4.8 Billiard cosmology with matter

To conclude this section we will apply the relation between physical branes and imaginary roots to billiard cosmology. In section 2.2 the matter component of the universe was provided by the fluxes. In this section we will add physical Kaluza-Klein particles and branes. In the absence of fluxes, we must make sure that the total charge of any type must be zero. We can do that by adding an equal amount of branes and anti-branes. As we have discussed in section 4.1, the presence of fluxes can induce a brane charge. Let α and β be two real roots such that $\gamma = \alpha + \beta$ is an imaginary root that is $\hat{W}$-dual to δ [defined in equation (2.3)]. If we turn on N_α units of flux corresponding to α and N_β units of the β-flux, then we get effective $N_\alpha N_\beta$ units of γ-charge which must be canceled by $N_\alpha N_\beta$ γ-type anti-branes. The flux contributes a term of the form

$$-\pi N_\alpha^2 e^{2\langle\alpha,\vec{h}\rangle} - \pi N_\beta^2 e^{2\langle\beta,\vec{h}\rangle}$$

to the effective lagrangian (2.15). N_γ branes (or anti-branes) of type corresponding to γ contribute a term of the form

$$-2\pi N_\gamma e^{\langle\gamma,\vec{h}\rangle} \tag{4.10}$$

to the lagrangian.

In [26, 27, 28] it was argued that each potential term $\exp\{2\langle\alpha,\vec{h}\rangle\}$ can be approximated by a wall in $\vec{h}$-space, and it was further argued that only the walls corresponding to the simple roots $\alpha = \alpha_{-1}, \ldots, \alpha_8$ are important. Up to a finite piece, the other walls are generically hidden behind the walls of the simple roots.

The new terms $\exp\langle\gamma,\vec{h}\rangle$ that come from matter correspond to potential terms that are in general smaller than the terms related to the simple roots. If $N_\gamma = |N_\alpha N_\beta|$, which is the minimal amount of branes necessary to balance the effective charge of the fluxes, then

$$2\pi N_\gamma e^{\langle\gamma,\vec{h}\rangle} \leq \pi N_\alpha^2 e^{2\langle\alpha,\vec{h}\rangle} + \pi N_\beta^2 e^{2\langle\beta,\vec{h}\rangle}\,.$$

In principle, however, we can let N_γ be larger if we add pairs of branes and anti-branes. In this case, since γ is lightlike, the term $\exp\langle\gamma,\vec{h}\rangle$ cannot be replaced by a wall because the billiard ball can penetrate the region where $\exp\langle\gamma,\vec{h}\rangle$ is large, as can be seen after writing down the equations of motion. (What makes this possible is the fact that the kinetic term in the lagrangian (2.15) is not positive definite.)

In addition, the dynamics could be more complicated since the branes could interact and annihilate. This topic is beyond the scope of this paper. (See [74] for a discussion on the dynamics of strings and branes in cosmology.)

It is also interesting to compare the term (4.10) to the effective σ-model proposed in [28]. There, an effective potential which contained a sum over all positive roots with terms of the form $\exp\{2\langle\gamma,\vec{h}\rangle\}$ was proposed to describe M-theory near a spatial Kasner-like singularity. Our term (4.10) is different by a factor of 2 in the exponent!

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5. Interactions

We have seen that real roots of E_{10} describe fluxes and instantons, and certain imaginary roots describe branes. In this section we will discuss combinations of roots. We will begin with a combination of two imaginary roots, and ask how the features of the interactions of the corresponding branes are related to the algebraic properties of the roots. We will then study the effects of a flux corresponding to a real root on a brane corresponding to an imaginary root.

5.1 Brane interactions

Take two imaginary roots α and β that correspond to physical branes as above. What can we say about the interaction between the branes from the algebraic perspective?

The inner product $(\alpha|\beta)$ encodes the basic properties of the interaction. We have discussed in section 3.2 the relation between threshold binding of instantons and the orthogonality of their corresponding real roots. We can now ask what is the condition on two imaginary roots α and β so that the corresponding physical branes could bind at threshold. Let M_α and M_β be the masses (i.e. actions per unit time) of the individual branes. The type of interaction we are interested in is characterized by the formation of a bound state with mass $M_\alpha + M_\beta$, in the absence of θ-angles. Take for example,

$$S_\alpha = 2\pi M_p^{12} V_{10} R_1 R_2 \,, \qquad S_\beta = 2\pi M_p^{12} V_{10} R_3 R_4 .$$

In the absence of θ-angles, the corresponding M2-branes can bind at threshold to form an object with mass $M_p^3(R_1R_2 + R_3R_4)$. We calculate $(\alpha|\beta) = -2$.

We conclude that the condition for binding at threshold is

$$(\alpha|\beta) = -2 \Longrightarrow \text{binding at threshold.} \tag{5.1}$$

This condition also applies for U-dual examples, such as a Kaluza-Klein particle with mass R_1^{-1} binding to an M2-brane with mass $M_p^3 R_1 R_2$, and so on. In particular, the fact that an M2-brane can end on an M5-brane [64] can be traced back to the possibility of the two objects to bind at threshold [57]. For this example, take an M2-brane with mass $M_p^3 R_1 R_2$ and an M5-brane with mass $M_p^6 R_2 R_3 R_4 R_5 R_6$; condition (5.1) is again satisfied.

The next type of interaction is typically characterized by forming a bound state with mass $\sqrt{M_\alpha^2 + M_\beta^2}$. For example, take α, β with

$$S_\alpha = 2\pi M_p^{12} V_{10} R_1 R_2 \,, \qquad S_\beta = 2\pi M_p^{12} V_{10} R_1 R_3 \,.$$

This corresponds to one M2-brane with mass $M_p^3 R_1 R_2$ and a second with mass $M_p^3 R_1 R_3$. The corresponding M2-branes can bind to form an object with mass $M_p^3 R_1 \sqrt{R_2^2 + R_3^2}$, according to the Pythagorean theorem.

This type of interaction also occurs when a brane absorbs a Kaluza-Klein particle and gains momentum in an orthogonal direction. For example, take α, β with

$$S_\alpha = 2\pi M_p^9 V_{10} R_1^{-1} \,, \qquad S_\beta = 2\pi M_p^{12} V_{10} R_2 R_3 \,.$$

Here α corresponds to Kaluza-Klein momentum in the 1^{st} direction, and β corresponds to an M2-brane in the 2^{nd} and 3^{rd} directions. The M2-brane can absorb the momentum and get an energy of $\sqrt{(M_p^3 R_2 R_3)^2 + (R_1^{-1})^2}$.

A third example is furnished by an M5-brane absorbing an M2-brane which becomes a 3-form tensor flux supported on its world-volume. In this case:

$$S_\alpha = 2\pi M_p^{12} V_{10} R_1 R_2 \,, \qquad S_\beta = 2\pi M_p^{15} V_{10} R_1 R_2 R_3 R_4 R_5 \,.$$

This is also dual to D-branes with electric or magnetic fluxes [75, 76, 63]. Inspired by the first example, we will refer to such an interaction as *Pythagorean binding.* In all these cases we have

$$(\alpha|\beta) = -1 \Longrightarrow \text{Pythagorean binding.}$$

In the case of Pythagorean interaction, either $\alpha - \beta$ or $\beta - \alpha$ is a positive real root.

For a third type of interaction, consider the process of brane creation. Take, for example, the case of [48] with two M5-branes that pass through each other,

$$S_\alpha = 2\pi M_p^{15} V_{10} R_1 R_2 R_3 R_4 R_5 \,, \qquad S_\beta = 2\pi M_p^{15} V_{10} R_1 R_6 R_7 R_8 R_9 \,.$$

In this, or any of its U-dual versions, we get

$$(\alpha|\beta) = -4 \Longrightarrow \text{Brane creation process.}$$

We have covered the cases $(\alpha|\beta) = -1, -2, -4$. It would be interesting to find the physical interpretation of other cases.

5.2 Interactions of branes with fluxes

In the previous section we discussed the interaction of two branes associated to the imaginary roots α, β. In this section we will take α to be imaginary and β to be real.

We assume that the imaginary root α corresponds to a Minkowski brane $\mathcal{B}_\alpha$ and the real root β corresponds to a flux. (The instanton associated with β creates a jump in that flux.) In this section we study the interaction of the brane $\mathcal{B}_\alpha$ with the flux. We will again attempt to characterize it according to the inner product $(\alpha|\beta)$.

We will assume that $\vec{h}$, the vector of $(\log R_i)$'s, is in such an asymptotic range that the brane $\mathcal{B}_\alpha$ is described by low-energy field theory and that the effect of the flux can be treated perturbatively, and we will restrict the discussion to first order.

As a first example, use the 10^{th} direction to reduce from M-theory to type-IIA and take $\mathcal{B}_\alpha$ to be a D2-brane so that

$$e^{\langle \vec{h}, \alpha \rangle} = M_p^{12} V_{10} R_8 R_9 = \frac{{M_s}^{11} V_9}{{g_s}^3} R_8 R_9 \,. \tag{5.2}$$

Where we have introduced the type-IIA string scale $M_s \equiv M_p^{\frac{3}{2}} R_{10}^{\frac{1}{2}}$, and coupling constant $g_s \equiv (M_p R_{10})^{\frac{3}{2}}$.

The D2-brane is described by a U(1) super-Yang-Mills theory with field strength $F_{\mu\nu}$ ($\mu,\nu = 0,\dots,2$), 7 scalars ϕ^I ($I = 1,\dots,7$), and 8 Majorana fermions ψ^a ($a = 1,\dots,8$), with lagrangian

$$L_{2+1D} = \frac{1}{4g_s}\left[F_{\mu\nu}F^{\mu\nu} + \delta_{IJ}\partial_\mu\phi^I\partial^\mu\phi^J + \delta_{ab}\overline{\psi}^a\,\partial\!\!\!/\psi^b\right]. \tag{5.3}$$

The index a of the fermions corresponds to the spinor representation **8** of the R-symmetry group Spin(7) and the index I corresponds to the vector representation **7**.

Let us first take the flux to be an NSNS flux H_{123}. The corresponding instanton is an NS5-brane in directions $4,\dots,9$, so that

$$2\pi e^{\langle\vec{h},\beta\rangle} = S_\beta = \frac{2\pi {M_s}^6}{{g_s}^2}R_4\cdots R_9 = 2\pi M_p^6 R_4\cdots R_9\,.$$

Note that $(\alpha|\beta) = 0$. The effect of such a flux is to "pin" the brane [77] and add a mass term to Φ^1,Φ^2,Φ^3 and to the fermions. The term linear in the flux is a mass term proportional to $H_{123}\overline{\psi}\Gamma^{123}\psi$ where Γ^I ($I = 1,\dots,7$) are Dirac matrices of Spin(7). It is worthwhile noting that after a series of U-dualities and a Penrose limit, the mass term above can be traced [78] to the mass term in the lightcone string theory that describes pp-waves [79, 80].

For the second example, let us stay with the D2-brane but take the flux to be an NSNS H_{129}. Now the flux has one leg along the D2-brane. The effect is [81, 78, 82] a nonlocal deformation of 2+1D super-Yang-Mills theory to a dipole theory [83]. The first order deformation is proportional to

$$H_{129}F_{9\mu}(\Phi^{[1}\partial^\mu\Phi^{2]} + \overline{\psi}\Gamma^{12}\sigma^\mu\psi)\,,$$

where 9 is a direction on the brane, and σ^μ is a 2+1D Dirac matrix.

In this case

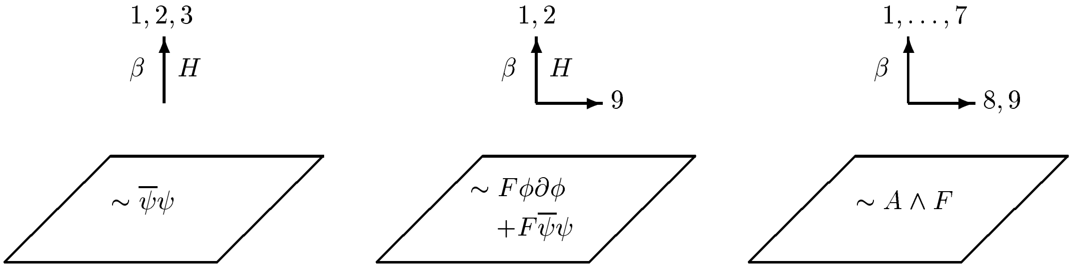
$$2\pi e^{\langle\vec{h},\beta\rangle} = S_\beta = 2\pi M_p^6 R_3 R_4\cdots R_8\,,$$

and $(\alpha|\beta) = -1$.

For a third example, we will add a Chern-Simons interaction to the lagrangian (5.3). To get the Chern-Simons interaction we will start with type-IIB this time. Take a D3-brane in directions $7,\dots,9$ with a low energy effective action given by $N = 4$ super-Yang-Mills theory, $L_{3+1D} = \frac{1}{4g_s}F_{\mu\nu}F^{\mu\nu} + \cdots$, and the scalars and fermions will not concern us this time. To add a Chern-Simons interaction we need to recall the coupling between the RR 0-form χ of type-IIB and the 2-form field strength F. It is

$$\int \chi F\wedge F = -\int d\chi\wedge A\wedge F\,.$$

So, if we can find an instanton that creates a constant gradient in χ in say the 7^{th} direction, we can get the Chern-Simons term $\int A\wedge F$ after dimensionally reducing to 2+1D, by forgetting the 7^{th} direction. (We also get a mass term for the fermions, as is required for supersymmetric Chern-Simons theory.) The flux $d\chi$ is created by a D7-brane instanton. But it is more convenient to formally T-dualize along the 7^{th} direction. The D3-brane



(a) Mass term $(\alpha|\beta) = 0$ (b) Dipole term $(\alpha|\beta) = -1$ (c) Chern-Simons term $(\alpha|\beta) = -2$

Figure 3: Three types of interactions of a D2-brane with flux. The D2-brane is in the plane of the $8^{th}, 9^{th}$ directions (and time). The imaginary root associated with it is α. The flux is associated with the real root β. The arrows indicate the directions of the flux: (a) A mass term appears as a result of an NSNS flux orthogonal to the brane; (b) A dipole interaction appears as a result of an NSNS flux with two legs orthogonal to the brane and one leg parallel to the brane; (c) A Chern-Simons term appears in massive type-IIA theory (the flux permeates throughout space).

becomes a D2-brane corresponding to the root α as before (5.2). The D7-brane instanton becomes a D8-brane with formal action,

$$2\pi e^{\langle \vec{h}, \beta \rangle} = S_\beta = \frac{2\pi {M_s}^9}{g_s} R_1 \cdots R_9 = 2\pi M_p^{12} R_1 \cdots R_9 R_{10}^3 \,.$$

Note that $(\alpha|\beta) = -2$. The D8-brane instanton turns type-IIA into a massive type-IIA [84, 85, 86], and we can arrive at the same Chern-Simons term by studying D-branes in massive type-IIA theory [87].

To conclude, we have found the following interactions of fluxes with branes (see figure 3),

$$\begin{aligned} (\alpha|\beta) &= 0 \Longrightarrow \text{Mass term}, \\ (\alpha|\beta) &= -1 \Longrightarrow \text{Dipole interaction}, \\ (\alpha|\beta) &= -2 \Longrightarrow \text{Chern-Simons}. \end{aligned} \tag{5.4}$$

5.3 Interaction potentials

How can we connect the interactions on the righthand column of (5.4) with the algebraic properties of the roots?

In this subsection we will write down formulas for the potential energies of the interactions. The formulas are in the spirit of the mass formula (4.6) and relate the derivative of the action with respect to conformal time $\tilde{\tau}$ to the roots.

Let us start with the case of a D2-brane in the $8^{th}, 9^{th}$ directions that is immersed in H_{123} NSNS flux, as in figure 3a. We need to calculate the mass term that is generated on the D2-brane world-volume. The magnitude of the flux is $H_{123}/R_1R_2R_3$, and it therefore follows that the mass term is proportional to $\exp \langle \alpha, \vec{h} \rangle$, in units dual to conformal time.

However, this formula does not tell us which degrees of freedom on the brane (i.e. which components of the fermions) receive a mass term. In order to distinguish the components, it will be more convenient to work with a mass term that preserves some supersymmetry. This can be achieved by adding H_{145} so that we now have both H_{123} and H_{145} fluxes perpendicular to the brane. The magnitudes of the fluxes $H_{123}/R_1R_2R_3$ and $H_{145}/R_1R_4R_5$ must be equal for some supersymmetry to be preserved. Since the mass term is supersymmetric (both fermions and bosons get the same mass), the ground state energy will not change. It will be simply the mass of the D2-brane. We need to find some way to coax the mass term to show itself as a change in energy.

We have at our disposal the option to add more branes and fluxes, and this is what we will do. We will adopt the same method used in [10]–[12]. We first note that the details of the H_{123}-related mass term are such that every state on the brane with angular momentum in the $2-3$ plane (which corresponds to an R-symmetry generator in the field theory on the D2-brane) gets an additional energy proportional to the angular momentum. Similarly, the H_{145} term is related to angular momentum in the $4-5$ plane. The ground state, having zero angular momentum, is not lifted.

Thus, to test the mass term we need to add angular momentum in the $2-3$ plane, say, and check the extra term in the energy of the state. But from the E_{10} perspective we can only easily add Kaluza-Klein momentum in some direction $1,\ldots,7$ perpendicular to the brane, not angular momentum. We need to find a trick to convert angular momentum to ordinary Kaluza-Klein momentum.

The trick is to add a "spectator" Kaluza-Klein monopole. Consider a type-IIA Kaluza-Klein monopole in $\mathbb{R}^{8,1}\times S^1$ space, with S^1 corresponding to the 5^{th} direction and let the monopole be extended in the $1^{st}, 6^{th},\ldots,9^{th}$ directions. We will ignore the $1^{st}, 6^{th},\ldots,9^{th}$ directions, for the moment. The Taub-NUT solution, corresponding to the Kaluza-Klein monopole, is

$$ds^2 = R_5^2 U\left(dx_5 - \sum_{i=2}^{4} A_i dx_i\right)^2 + U^{-1}\sum_{i=2}^{4} dx_i^2, \quad 0\le x_5\le 2\pi, \quad U\equiv\left(1+\frac{R_5}{\sqrt{\sum_{i=2}^4 x_i^2}}\right)^{-1},$$

where A_i is the gauge field of a monopole centered at the origin. The Taub-NUT solution is a fibration of a circle (the 5^{th} direction) over $\mathbb{R}^3$ such that at ∞ the circle has a constant radius R_5. The Taub-NUT solution is smooth at the origin. The relevant point for us is that there is an isometry that looks like a translation in the 5^{th} (the S^1's) direction at ∞ and as a rotation in SO(4) of the $\mathbb{R}^4$ tangent space at the origin. If we place a D-brane at the origin (and allow it to extend in some of the other directions $1,6,\ldots,9$) we can convert angular momentum in directions $2,3,4,5$ perpendicular to the brane to Kaluza-Klein momentum in the 5^{th} direction at ∞ far from the brane. The upshot is that together with the Kaluza-Klein monopole, states with Kaluza-Klein momentum in the 5^{th} direction should get extra energy.

Now let us rephrase the story in E_{10} language. First, it will be convenient to generate the NSNS fluxes H_{123} and H_{145} not via an instanton, as we did in section 5.2, but via a θ-angle. The spectator Kaluza-Klein monopole helps us with that too [81]. Suppose that far away from the origin we try to set up a constant NSNS B_{15}-field. The Taub-NUT geometry looks locally like $\mathbb{R}^3 \times S^1$ with a constant S^1, so unless we get very close to the origin there is no problem in setting up the constant B_{15}-field. But if try to extend B_{15} to the full Taub-NUT geometry we run into an obstacle. We have to set the B-field to be proportional to the global angular 1-form of the fibration, but that form is not closed, so there has to be an $H = dB$ flux. In fact, at the origin the value of the flux turns out to be $|H|^2 \sim (B_{15}/R_1 R_5^2)^2$, where $0 \leq B_{15} < 2\pi$ is the asymptotic value of the B-field at infinity [81].

Now we are ready to translate to E_{10}-roots. Let us lift back from type-IIA to M-theory along the 10^{th} direction. We have the imaginary root α that corresponds to the D2-brane, the imaginary root σ that corresponds to the spectator Kaluza-Klein monopole, the imaginary root γ that corresponds to Kaluza-Klein momentum in the 5^{th} direction (to be converted to angular momentum by the Kaluza-Klein monopole) and finally, we have the real root η that corresponds to the θ-angle that is the NSNS 2-form flux $\mathcal{C}_\eta \equiv B_{15}$. The corresponding actions are listed in the following table:

Object	Root	Action/2π
D2-brane	α	$e^{\langle\alpha,\vec{h}\rangle} = M_p^{12} V_{10} R_8 R_9$
Spectator	σ	$e^{\langle\sigma,\vec{h}\rangle} = M_p^{18} V_{10}^2 R_5/R_2 R_3 R_4$
Momentum	γ	$e^{\langle\gamma,\vec{h}\rangle} = M_p^9 V_{10}/R_5$
B-flux	η	$e^{\langle\eta,\vec{h}\rangle} = M_p^3 R_1 R_5 R_{10}$

The extra energy due to the interaction of the flux with the brane that we expect is

$$\Delta\mathcal{V} \sim \frac{\mathcal{C}_\eta}{{M_s}^2 R_1 R_5^2} = \frac{\mathcal{C}_\eta}{M_p^3 R_1 R_5^2 R_{10}}.$$

In the spirit of the mass formula (4.6), we write it as

$$\frac{d\tilde{S}_I}{d\tilde{\tau}} = 2\pi M_p^9 V_{10} \Delta\mathcal{V} \sim e^{\langle\gamma-\eta,\vec{h}\rangle}\mathcal{C}_\eta\,, \qquad (\gamma|\eta) = -1. \tag{5.5}$$

where $\tilde{S}_I$ is the extra term in the action due to the interaction. We see that the spectator root σ does not enter into the interaction formula. Note also that

$$(\alpha|\gamma) = -1\,, \qquad (\alpha|\eta) = -1\,.$$

We can similarly study the case depicted in figure 3*b*. In this case, states with $2-3$ or $4-5$ angular momentum have an effective electric dipole on the D2-brane worldvolume. The dipole vector is proportional to the angular momentum and is directed along the 9^{th} direction. To probe it we need to add an additional electric field on the D2-brane, as was done in [81]. We can do it by adding an extra fundamental string charge to the setting, but we will not do that here.

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6. A note on supersymmetry

For M-theory on T^8, each instanton that corresponds to a (real) positive root of $E_{8(8)}$ breaks half the supersymmetry. The supersymmetry generators transform in the vector representation of the double cover of the maximal compact subgroup $K_8 = \mathrm{Spin}(16)/\mathbb{Z}_2$ of $E_{8(8)}$.[1] (Spin(16) has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ center. The first $\mathbb{Z}_2$ factor is trivial in the spinor representation, while the second factor is trivial in the vector representation. The $\mathbb{Z}_2$ factor in K_8 is the second one, since the adjoint representation of E_8 decomposes as the **120** adjoint of $so(16)$ plus the **128** spinor of $so(16)$, but does not contain the vector **16**.)

Physically, each real positive root therefore defines a subalgebra of the Lie algebra $so(16)$. This is the subalgebra that preserves the unbroken supersymmetry. We will now study this relation from the group theoretic point of view. For a related discussion see [88, 18, 19, 20, 24].

First, let us explain what we mean by the action of K. Classically, the low-energy limit of M-theory on T^8 is described by a supersymmetric σ-model with target space G/K where $G = E_{8(8)}$ and $K = \mathrm{Spin}(16)/\mathbb{Z}_2$. A point in the target space can be parameterized as a coset gK with $g \in G$. The σ-model can be formulated as a gauged σ-model with target space G and gauge group K acting on G from the right. The fermions ψ are in the vector representation of K. But this action of K from the right is not physically interesting, because it is merely a gauge symmetry. We are interested in the *gauge invariant* combinations $g\psi$ on which K acts from the *left* as $g\psi \mapsto xg\psi$, for $x \in K$. (Note that $g\psi$ is defined only up to a sign ambiguity because of the $\mathbb{Z}_2$ factor in K, but bilinears in ψ are well defined.) This K-symmetry, being broken by instantons, is not a good quantum symmetry. But this is precisely the point here — each instanton term breaks a part of K and defines an unbroken subgroup.

Consider 10+1D uncompactified M-theory. The supersymmetry generators are Majorana spinors of $so(10,1)$. Under $so(8) \oplus so(2,1) \subset so(10,1)$ they decompose as $\mathbf{32} = (\mathbf{8}_c \oplus \mathbf{8}_s) \otimes \mathbf{2}$, where $\mathbf{8}_c$ and $\mathbf{8}_s$ are the two real spin representations of $so(8)$ and $\mathbf{2}$ is the real spin representation of $so(2,1)$. Let V be the 16-dimensional real vector space $\mathbf{8}_c \oplus \mathbf{8}_s$. Both $\mathbf{8}_c$ and $\mathbf{8}_s$ have an $so(8)$-invariant bilinear form that we denote by $(\cdot|\cdot)_c$ and $(\cdot|\cdot)_s$, respectively. The supersymmetry generators are 2+1D spinors which take values in V. The R-symmetry algebra $so(16)$ acts on V as the subset of $gl(V,\mathbb{R})$ that preserves the bilinear form $(\cdot|\cdot)_c + (\cdot|\cdot)_s$. Choose a Majorana representation so that the Dirac Γ-matrices are purely imaginary. A massless particle with momentum $\vec{p} = (p_0, p_1, \ldots, p_{10})$ preserves the supersymmetry generators that satisfy $(p_0\Gamma^0 + \sum_1^{10} p_i\Gamma^i)x = 0$, for $x \in V$. Now consider a Kaluza-Klein instanton with momentum in the j^{th} direction and world-line in the k^{th} direction, so that the action is $R_k R_j^{-1}$. After Wick rotating the massless particle we find that the instanton preserves $x \in V$ that commute with Γ^{kj}. It defines the subspace

$$W_{k;j} \stackrel{\text{def}}{=} \{x \in V : i\Gamma^{kj}x = x\} \subset V\,.$$

[1] We are grateful to R. Borcherds for pointing out to us that K_8 should not be denoted by SO(16).

We denote the subalgebra of $so(16)$ that preserves $W_{k;j}$ by

$$U_{k;j} \stackrel{\text{def}}{=} \{g \in so(16) : \Gamma^{kj} g = g\Gamma^{kj}\} \subset so(16)\,.$$

$U_{k;j}$ is isomorphic to $u(8)$.

Similarly, an M2-brane instanton stretched in directions j_1, j_2, j_3 defines a subspace

$$W_{j_1 j_2 j_3} \stackrel{\text{def}}{=} \{x \in V : i\Gamma^{j_1 j_2 j_3} x = x\} \subset V\,.$$

We denote the subalgebra of $so(16)$ that preserves $W_{j_1 j_2 j_3}$ by

$$U_{j_1 j_2 j_3} \stackrel{\text{def}}{=} \{g \in so(16) : \Gamma^{j_1 j_2 j_3} g = g\Gamma^{j_1 j_2 j_3}\} \subset so(16)$$

$U_{j_1 j_2 j_3}$ is also isomorphic to $u(8)$.

To see this, let us take, without loss of generality, $(j_1, j_2, j_3) = (1, 2, 3)$, and let us decompose the representation V of $so(8)$ under the Lie algebra $so(3) \oplus so(5) \subset so(8)$. We find that V decomposes as the complex representation $(\mathbf{2}, \mathbf{4})$ of $so(3) \oplus so(5)$. The components of a vector $z \in V$ can be written as $z^{\alpha a}$ where $\alpha = 1, 2$ and $a = 1, \ldots, 4$. The bilinear form $(\cdot|\cdot) \equiv (\cdot|\cdot)_c + (\cdot|\cdot)_s$ can be written as $(z|z) = \sum_{\alpha,a} |z^{\alpha a}|^2$. We can decompose each component into its real and imaginary parts as $z^{\alpha a} = u^{(\alpha a)} + iv^{(\alpha a)}$. The elements of $so(3)$ and $so(5)$ mix the components $u^{(\alpha a)}$ with $v^{(\alpha a)}$ and $i\Gamma^{123}$ acts as $z^{\alpha a} \to iz^{\alpha a}$ and therefore as $u^{(\alpha a)} \to -v^{(\alpha a)}$, and $v^{(\alpha a)} \to u^{(\alpha a)}$. The subalgebra $U_{j_1 j_2 j_3} \subset so(16)$ is therefore the subalgebra that commutes with the transformation above and is isomorphic to $u(8)$, as we claimed.

On the other hand, as we have reviewed in section 2.3, the Kaluza-Klein and the M2-brane instantons correspond to positive roots of E_8. Thus, in the same way as above, every positive root α of E_8 defines a subalgebra U_α of $so(16)$.

The subalgebra $so(16) \subset E_8$ is generated by $e_i - f_i$ $(i = 1, \ldots, 8)$, where e_i, f_i are Chevalley generators as in (2.1.1). Let u be a generator of the 1-dimensional root space $\dot{\mathbf{g}}_\alpha \subset E_8$. The compact involution on E_8 is defined by the generating relations

$$\omega(e_i) = -f_i\,, \qquad \omega(f_i) = -e_i\,, \qquad \omega(h_i) = -h_i\,. \tag{6.1}$$

Define $q_\alpha \stackrel{\text{def}}{=} u + \omega(u)$. Then U_α is the subalgebra of $so(16)$ that commutes with q_α.

To see this consider the subsets

$$U_{j_1;j_2} \stackrel{\text{def}}{=} \{g \in so(16) : \Gamma^{j_1 j_2} g = g\Gamma^{j_1 j_2}\} \subset so(16)\,, \qquad 1 \le j_1 < j_2 \le 8\,.$$

corresponding to Kaluza-Klein particles. Note that if we drop the 8^{th} node of the Dynkin diagram of E_8, we get the subalgebra $sl(8) \subset E_8$. This subalgebra is generated by e_i, f_i, h_i for $i = 1, \ldots, 7$. The combinations $e_i - f_i$ for $i = 1, \ldots, 7$ generate $so(8) \subset sl(8)$. The matrix $\Gamma^{j_1 j_2}$ can be identified with an $so(8)$ generator on the spinor representation V. It is easy to see that for $j_1 = i$, and $j_2 = i+1$, this generator can be identified with $e_i - f_i$. Thus $U_{j_1 j_2} \cap so(8) \subset so(16)$ is the subgroup that commutes with $e_i - f_i$. It is also not hard to see that the 8^{th} generator $e_8 - f_8$ can be identified with $i\Gamma^{678}$ acting on V. The statement that $U_{i;i+1}$ is the subspace of $so(16)$ that commutes with $e_i - f_i$ (for $i = 1, \ldots, 7$) follows.

These constructions can be extended to the infinite dimensional Lie algebras $\mathbf{g} = E_9$ and $\hat{\mathbf{g}} = E_{10}$. For $\mathbf{g}$, the algebra $\mathbf{k}$ is defined as the ω-invariant subalgebra of $\mathbf{g}$. It is denoted by $so(16)^\infty$ [8] and is *not* to be confused with the affine $\widehat{so}(16)$ Lie algebra.[2] Any root α of E_9 defines the subalgebra

$$U_\alpha \stackrel{\text{def}}{=} \{v \in \mathbf{k} : [v, u + \omega(u)] = 0 \ \forall u \in \mathbf{g}_\alpha\} \subset \mathbf{k}$$

where $\mathbf{g}_\alpha$ is the root space of α, which could now be of dimension higher than 1 if α is an imaginary root.

For a real root α, one can argue that $U_\alpha \sim su(8)^\infty \oplus u(1)$, where $su(8)^\infty$ is constructed from the affine Lie algebra $\hat{E}_7$ in a similar way to the construction of $so(16)^\infty$ from the affine Lie algebra $\mathbf{g} = \hat{E}_8$, that is, by considering the generators that are invariant under an involution.[2] (To see this, take $\alpha = \alpha_{-1}$ without loss of generality, and note that the only elements of the form $e_\beta - f_\beta$ that commute with $e_\alpha - f_\alpha$ are such that $\beta = n\delta + \beta'$ with β' a root of $E_7 \subset E_8 \subset E_9$.)

For imaginary roots $\alpha = n\delta$ it is also not hard to see that $U_{n\delta}$ is trivial.

It would be interesting to find a nontrivial extension of the definition of U_α for imaginary roots and to explore its relationship with its associated brane. Perhaps, one needs to find an element u of the root space $\mathbf{g}_\alpha$, such that the centralizer of $u + \omega(u)$ (i.e. all $v \in \mathbf{k}$ such that $[u + \omega(u), v] = 0$) is maximal in some sense. We will leave this for future work.

7. Constructing a hamiltonian

It is time to collect all the pieces into one framework. In this section we will construct a hamiltonian, based on E_{10}, that describes some of the features of M-theory on T^{10}, that we discussed above. As we do not know the full details, we will only present a few simple observations.

The Lie algebra E_{10} is integrable, which means that the Lie group $G_{10} \equiv \exp E_{10}$ can be defined. We can also define $K_{10} \equiv \exp \hat{\mathbf{k}}$. (Here $\hat{\mathbf{k}}$ is defined to be the ω-invariant subalgebra of E_{10}, where ω is the compact involution, defined similarly to (6.1).) K_{10} is not actually compact but seems to be what we need (see [8]). A natural starting point is a 0+1D (quantum mechanics) σ-model on the coset space G_{10}/K_{10}. The G_{10} invariant metric on G_{10}/K_{10} is "almost" positive definite. To see what this means, consider the metric on the Lie algebra E_{10}. The Lie algebra has an invariant bilinear form [41], but it is not positive definite. This form turns out to be negative definite in the directions of $\hat{\mathbf{k}}$. In addition, the Cartan subalgebra $\hat{\mathfrak{h}}_{\mathbb{R}}$ has signature $(9,1)$ which means that it has a negative-norm element x. However, when restricted to the subspace orthogonal to x and $\hat{\mathbf{k}}$, the invariant bilinear form is positive definite. (See theorem 11.7 of [41] for more details.) Thus, after modding out G_{10} by K_{10} we get rid of all the negative-norm directions except the one in the Cartan subalgebra.[3]

[2] We are grateful to A. Keurentjes for pointing out an incorrect statement we had made in a previous version.

[3] We are grateful to Edward Witten for raising this issue.

We now take the effective hamiltonian to be proportional to the laplacian $\mathcal{H} = -\triangle$ on the infinite dimensional space G_{10}/K_{10}. The wave functions are required to satisfy a generalized Wheeler-DeWitt equation

$$\mathcal{H}\Psi \equiv -\triangle\Psi = 0\,. \tag{7.1}$$

The manifest G_{10} invariance could be spontaneously broken to the U-duality subgroup $E_{10}(\mathbb{Z})$ by requiring Ψ to be only $E_{10}(\mathbb{Z})$ invariant. This can be implemented by defining the target space, on which the laplacian $\triangle$ acts, as the coset $E_{10}(\mathbb{Z})\backslash G_{10}/K_{10}$.

Before we proceed, we have to mention that equation (7.1) appeared in similar contexts before. A σ-model on the same coset space G_{10}/K_{10} was presented in [25] and an extension to G_{11}/K_{11} was presented in [13, 15, 22]. Furthermore, equation (7.1) was also proposed in [10].

The new point of this paper is that we can identify a mechanism to go beyond dimensionally reduced classical supergravity and to test the approximation (7.1) quantum mechanically. We will attempt a quantum mechanical treatment of the variables of G_{10}/K_{10} associated with imaginary roots, and we will compare the resulting energy levels to the energies of branes and Kaluza-Klein particles that can be introduced into the evolving universe.

Specifically, "excited states" of the universe, with branes or Kaluza-Klein particles, appear to be related to excited Landau levels of a certain effective magnetic field that is naturally generated inside $E_{10}(\mathbb{Z})\backslash G_{10}/K_{10}$ when the canonical momenta dual to the variables associated with imaginary roots are nonzero. The separation between Landau levels roughly matches the expected energies of the branes, but unfortunately there is a mismatch by a factor of 2π. There are also a few other puzzles, related to the zero-point energy and to total neutrality. We present the ideas here anyway, in the hope that there might be some way to "fix" the problems. Let us now construct the model.

7.1 The variables

The model is a 0+1D quantum mechanics. "Time" is taken to be M-theory's conformal time defined in (2.14).

Skipping the proof, which can be found elsewhere (see [28] and also [47] for the finite dimensional case), the variables of the coset G_{10}/K_{10} can be described as follows. We have 10 real variables, each taking values in $\mathbb{R}$, given by the components of $\vec{h}$ that are related to the physical radii as in (2.13). In addition, we have an infinite tower of periodic variables with period 2π; there is one variable $\mathcal{C}_\alpha$ associated with every *positive real root* α of E_{10}, and there are $\mathrm{mult}(\gamma)$ variables $\mathcal{C}_{\gamma,j}$ $(j = 1, \ldots, \mathrm{mult}(\gamma))$ associated with any positive imaginary root γ of E_{10}. Here $\mathrm{mult}(\gamma)$ is the multiplicity of the root γ. Occasionally, it will be convenient to suppress the index j. In that case, it will be understood that $\mathcal{C}_\gamma$ denotes some linear combination of the $\mathcal{C}_{\gamma,j}$'s.

We will now construct the hamiltonian. It is going to be convenient to identify the charges of the variables under the $\mathbb{R}^{10}$ Cartan subalgebra $\hat{\mathfrak{h}}_{\mathbb{R}}$ of E_{10} that acts as

$$\vec{h} \mapsto \vec{h} + \vec{\epsilon}\,, \qquad \vec{\epsilon} \in \hat{\mathfrak{h}}_{\mathbb{R}}\,.$$

Under this symmetry

$$\mathcal{C}_\alpha \mapsto e^{\langle\alpha,\vec{\epsilon}\rangle}\mathcal{C}_\alpha\,, \qquad \mathcal{C}_{\gamma,j} \mapsto e^{\langle\gamma,\vec{\epsilon}\rangle}\mathcal{C}_{\gamma,j}\,. \tag{7.2}$$

This symmetry does not preserve the periodicity of $\mathcal{C}_\alpha, \mathcal{C}_{\gamma,j}$, but it is a symmetry of the hamiltonian.

The hamiltonian is constructed from functions of $\vec{h}, \mathcal{C}_\alpha, \mathcal{C}_{\gamma,j}$ and their first derivatives $\partial/\partial h_i$, $\partial/\partial\mathcal{C}_\alpha$, $\partial/\partial\mathcal{C}_{\gamma,j}$. It is probable that we also need to include fermionic degrees of freedom, but we will completely ignore the fermions in this section, for simplicity.

7.2 The hamiltonian

The hamiltonian $\mathcal{H}$ preserves G_{10}, and hence all the terms appearing in it conserve the $\mathbb{R}^{10}$ charges of (7.2). Up to a factor of -1, it is the E_{10}-invariant laplacian $\mathcal{H} = -\triangle$. Explicitly, it contains the following terms.

First, there is a term that contains only $\vec{h}$ and is given by

$$\mathcal{H}_h \stackrel{\text{def}}{=} -\frac{1}{8\pi}\left[\sum_{k=1}^{10}\frac{\partial^2}{\partial {h_k}^2} - \frac{1}{9}\left(\sum_{k=1}^{10}\frac{\partial}{\partial h_k}\right)^2 + \sum_{k=1}^{10}\left(2k-\frac{56}{3}\right)\frac{\partial}{\partial h_k}\right]. \tag{7.3}$$

The linear term might appear strange, but it can be deduced by extrapolation from the E_8 case. [It is also required in order for the instanton terms $\exp(-S_{\alpha_k} + i\mathcal{C}_{\alpha_k})$ to be harmonic functions for the simple roots α_k. See section 2.3.] The apparent S_{10}-asymmetric form of (7.3) is also not a problem since the decomposition into positive and negative roots already breaks this S_{10} permutation symmetry.

Then we have the terms

$$\mathcal{H}_0 \stackrel{\text{def}}{=} -\pi\sum_{\alpha^2=2} e^{2\langle\alpha,\vec{h}\rangle}\frac{\partial^2}{\partial{\mathcal{C}_\alpha}^2} - \pi\sum_{\alpha^2\le 0}\sum_{j=1}^{\text{mult}(\alpha)} e^{2\langle\alpha,\vec{h}\rangle}\frac{\partial^2}{\partial{\mathcal{C}_{\alpha,j}}^2}$$

Note that this term is invariant under (7.2) as $\partial/\partial\mathcal{C}_\alpha$ has charge $-\alpha$. (The factors of π appear because of our choice of periodicity of $\mathcal{C}_\alpha$.)

In addition to $\mathcal{H}_h$ and $\mathcal{H}_0$ we have an infinite series of ever more complex terms, so that

$$\mathcal{H} = \mathcal{H}_h + \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \cdots$$

where $\mathcal{H}_n$ is quadratic in $\partial/\partial\mathcal{C}_\alpha$ (or $\partial/\partial\mathcal{C}_{\gamma,j}$) but is a polynomial of degree n in $\mathcal{C}_\alpha$ (or $\partial/\partial\mathcal{C}_{\gamma,j}$). The first terms look schematically like

$$\mathcal{H}_1 \sim \sum_{\gamma=\alpha+\beta} e^{\langle\beta+\gamma-\alpha,\vec{h}\rangle}\mathcal{C}_\alpha\frac{\partial^2}{\partial\mathcal{C}_\beta\partial\mathcal{C}_\gamma} = \sum_{\gamma=\alpha+\beta} e^{2\langle\beta,\vec{h}\rangle}\mathcal{C}_\alpha\frac{\partial^2}{\partial\mathcal{C}_\beta\partial\mathcal{C}_\gamma}\,. \tag{7.4}$$

Note that the dependence on $\vec{h}$ is entirely determined by conservation of $\mathbb{R}^{10}$-charge. The expression $\mathcal{H}_1$ can be deduced from the E_{10} transformation properties of $\mathcal{H}_0$ and the invariance of the total expression $\mathcal{H}$. Similarly, each consecutive $\mathcal{H}_n$ can be deduced from the E_{10}-transformation properties of its predecessors. We will not do the explicit computation here. It can be found in [28] (see also [12]).

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The kinetic term $\mathcal{H}_0$ already contains all the "wall" potential terms required for billiard cosmology without branes (see section 2.2). A state with N_α units of the flux associated with the real root α has a wave function that behaves as $\sim \exp iN_\alpha\mathcal{C}_\alpha$. It is an eigenstate of $\partial/\partial\mathcal{C}_\alpha$. Setting $\partial/\partial\mathcal{C}_\alpha \to iN_\alpha$ in $\mathcal{H}_0$ we obtain the potential $\pi N_\alpha^2 \exp(2\langle\alpha, \vec{h}\rangle)$. That leads to the expressions discussed in section 2.2 for the wall potentials of billiard cosmology [25].

The term $\mathcal{H}_1$ is also interesting in that it tells us that the target space of the σ-model is not just a product $\mathbb{R}^{10} \times S^1 \times S^1 \times \cdots$ — with each S^1 corresponding to a different $\mathcal{C}_\alpha$ — but is a nontrivial circle bundle. For example, take two positive real roots α, β, such that $\gamma = \alpha + \beta$ is also real. Being periodic, the associated variables $\mathcal{C}_\alpha, \mathcal{C}_\beta$ parameterize a T^2. If $\gamma = \alpha + \beta$ and the commutator of the Lie algebra generators $[J^{+\alpha}, J^{+\beta}]$ is proportional to $J^{+\gamma}$ then the circle associated with the periodic variable $\mathcal{C}_\gamma$ is nontrivially fibered over the T^2. The first Chern class of the fibration is $c_1 = 1$. This is easily seen by noting that the infinitesimal E_{10} transformation $\exp(\epsilon J^{+\alpha})$ acts as

$$\mathcal{C}_\alpha \to \mathcal{C}_\alpha + \epsilon\,, \qquad \mathcal{C}_\gamma \to \mathcal{C}_\gamma + \frac{\epsilon}{2\pi}\mathcal{C}_\beta\,.$$

For real roots, this geometrical fact has some interesting physical consequences such as Wess-Zumino terms, but we will not discuss this here (see for instance [10, 28]). If γ is an imaginary root, we have to be careful because of its multiplicity. The statement is that if the commutator of the Lie algebra generators $[J^{+\alpha}, J^{+\beta}]$ is proportional to a generator $J^{+\gamma,j}$ in the root space $\hat{\mathbf{g}}_\gamma$, then $\mathcal{C}_{\gamma,j}$ is nontrivially fibered over T^2 with first Chern class $c_1 = 1$. This fact will be *crucial* in section 7.4.

To summarize, $\mathcal{H}$ is a quadratic differential operator (which also contains the linear term in $\mathcal{H}_h$). It is essentially determined by E_{10}-invariance.

7.3 Instanton effects

Universes with a flux $\mathcal{C}_\alpha$ turned on must have a wave-function of the form

$$\Psi_\alpha = e^{-S_\alpha + i\mathcal{C}_\alpha}\,(\cdots)\,. \tag{7.5}$$

where $(\cdots)$ is independent of $\mathcal{C}_\alpha$. The prefactor expresses the tunneling amplitude from a state without flux to a state with flux. A state with N_α units of flux is an eigenstate of $-i\partial/\partial\mathcal{C}_\alpha$. The action S_α is in general a complicated expression of the fluxes and of $\vec{h}$, but when all the fluxes (except $\mathcal{C}_\alpha$ of course) are set to zero, S_α reduces to $2\pi \exp\langle\alpha, \vec{h}\rangle$ — the simplified expression that we have been using throughout this paper. The fact that the prefactor $\exp\{-S_\alpha + i\mathcal{C}_\alpha\}$ is a harmonic function on G_{10}/K_{10} if α is a simple root (see section 2.3) is intriguing, but it seems that extra terms must be added to $\mathcal{H}$ in order for (7.5) to be an eigenfunction.

7.4 Branes and Landau levels

In section 4 we argued that a prime imaginary root γ with $\gamma^2 = 0$ corresponds to a Minkowski brane. We found a mass formula (4.6) that expresses the mass (defined with respect to conformal time) in terms of γ, as $2\pi \exp\langle\vec{h}, \gamma\rangle$. If there are n branes, we expect a contribution to the hamiltonian of the form $2\pi n \exp\langle\vec{h}, \gamma\rangle$. We will now suggest a way in which such a term could come from quantizing the variables $\mathcal{C}_{\gamma,j}$.

Decompose $\gamma = \alpha + \beta$ as a sum of two positive real roots, and suppose that the Lie algebra element corresponding to $\mathcal{C}_{\gamma,j}$ is proportional to the commutator $[J^{+\alpha}, J^{+\beta}]$. Then, $\mathcal{C}_{\gamma,j}$ is a local coordinate on a circle bundle over the $\mathcal{C}_\alpha, \mathcal{C}_\beta$ torus, as explained in section 7.2. Since the $\mathcal{C}_{\gamma,j}$-circle is nontrivially fibered over the $\mathcal{C}_\alpha, \mathcal{C}_\beta$ torus, it follows that the negative of the laplacian $\triangle$ contains terms of the form

$$\mathcal{H}' = -\pi e^{2\langle\vec{h},\alpha\rangle} \frac{\partial^2}{\partial {\mathcal{C}_\alpha}^2} - \pi e^{2\langle\vec{h},\beta\rangle} \left(\frac{\partial}{\partial \mathcal{C}_\beta} - \frac{\mathcal{C}_\alpha}{2\pi} \frac{\partial}{\partial \mathcal{C}_{\gamma,j}} \right)^2 - \pi e^{2\langle\vec{h},\gamma\rangle} \frac{\partial^2}{\partial {\mathcal{C}_{\gamma,j}}^2} \tag{7.6}$$

(The $\vec{h}$-dependent coefficients are determined by the $\mathbb{R}^{10}$-symmetry (7.2).)

Suppose we have a state Ψ for which $-i\partial/\partial\mathcal{C}_{\gamma,j} = N_{\gamma,j}$. Plugging that into (7.6) we find that $\mathcal{H}'$ describes the hamiltonian of an abstract charged particle on a torus (parameterized by the coordinates $\mathcal{C}_\alpha, \mathcal{C}_\beta$) with $N_{\gamma,j}$ units of magnetic flux. The "cyclotron" frequency is $\omega = \exp\langle\vec{h}, \alpha+\beta\rangle |N_{\gamma,j}| = \exp\langle\vec{h},\gamma\rangle |N_{\gamma,j}|$. Eliminating $\mathcal{C}_\alpha$ we get "Landau levels" with energy

$$\mathcal{H}'' = e^{\langle\vec{h},\gamma\rangle} |N_{\gamma,j}| \left(n + \frac{1}{2} \right) + \pi e^{2\langle\vec{h},\gamma\rangle} |N_{\gamma,j}|^2$$

It is now tempting to compare these states with states of the universe that contain n bound states of $N_{\gamma,j}$ branes. The n-dependent part of the energy is $n|N_{\gamma,j}| \exp\langle\vec{h},\gamma\rangle$. According to the mass formula (4.6) this is similar to the contribution of n bound states of $N_{\gamma,j}$ branes to the energy, but unfortunately there is a 2π mismatch.

The remaining terms in $\mathcal{H}''$, which include the zero-point energy and the $\mathcal{C}_{\gamma,j}$-flux contribution, are independent of n. We do not know how to interpret them, but perhaps they behave like a cosmological constant. Perhaps they can be cancelled if supersymmetry is properly taken into account. It might also be possible to consistently leave the problematic terms $\exp(2\langle\vec{h},\gamma\rangle)\partial^2/\partial\mathcal{C}_{\gamma,j}^2$ out of $\mathcal{H}$.

We also have to mention, however, that when all of space is compact we cannot add branes at will, because the total charge has to cancel. But we can add pairs of branes and anti-branes. If there are N_p pairs we expect a contribution to the hamiltonian of the form $2N_p \times 2\pi \exp\langle\vec{h},\gamma\rangle$. Furthermore, in section 4 we constructed branes from pairs of instantons corresponding to real roots α and β with $\gamma = \alpha+\beta$ such that $\gamma^2 = 0$. We argued that if there are N_α units of flux associated with the real root α and N_β units of flux associated with the real root β then, in the setting of section 4.1, there must also be a net number of $N_\alpha N_\beta$ branes, for charge neutrality. At the moment, we do not know how charge neutrality appears in the E_{10} formalism.

Finally, let us show that the mismatch factor of 2π between the Landau levels and the expected masses of branes is not an artifact of the conventions. To see this compare the energy levels of a (nonrelativistic) free particle on $T^2 = S^1 \times S^1$ with one unit of magnetic flux, to the energy levels of the same particle on the same T^2 without any magnetic flux. In the second case, the energy levels are of the form $C_1 n_1^2 + C_2 n_2^2$, where C_1, C_2 are constants and $n_1, n_2 \in \mathbb{Z}$. In the first case, the energy levels are $(2n+1)\sqrt{C_1 C_2}/2\pi$. The factor of 2π in the last formula is the source of the mismatch.

7.5 Summary

Some features of M-theory on T^{10} are effectively described by a harmonic function on the target space $E_{10}(\mathbb{Z})\backslash G_{10}/K_{10}$ satisfying

$$\triangle\Psi = 0\,.$$

The Wheeler-DeWitt wave-function Ψ is a sum of terms with different eigenvalues of the various fluxes $N_\alpha = -i\partial/\partial\mathcal{C}_\alpha$. The behavior of Ψ as a function of the radii (encoded in $\vec{h}$) crucially depends on N_α. Different pieces of Ψ therefore describe completely different evolutions of the universe and can thus be separated.

The term without fluxes (all $N_\alpha = 0$) describes possible Kasner evolutions with $\|\vec{p}\|^2 = 0$ (in the notation of section 2.2) and, according to [9], can never describe a classical universe in the far future. Terms in Ψ for which only the fluxes N_α that are associated to simple roots are nonzero describe a chaotic evolution as in [26] and are also never classical in the far future.

But terms with nonzero quantum numbers N_γ associated to imaginary roots can describe, as we suggested, universes with an ordinary matter component composed of Kaluza-Klein states, or branes. These universes can have a classical evolution in the far future (a "safe" region of moduli space, in the terminology of [9]). Unfortunately, the brane masses that we obtain are smaller than the correct masses by a factor of 2π.

8. Conclusions and discussion

The infinite dimensional Lie algebra E_{10} is likely to play an important role in a fundamental formulation of M-theory. Its roots encode the kinematic properties of branes. Real roots encode instanton actions, and, as we have proposed in this paper, certain imaginary roots correspond to branes. We have also seen that the inner product of two imaginary roots α, β encodes basic properties of the interaction between the two corresponding branes. We have interpreted the values $(\alpha|\beta) = -1, -2, -4$. Similarly, we have seen that the inner product of an imaginary root and a real root encodes the basic properties of the interactions of the corresponding brane with the corresponding flux. We have interpreted the values $(\alpha|\beta) = 0, -1, -2$.

We have begun to construct a Hilbert space and an effective hamiltonian that can describe some features of M-theory in this setting. We have argued that this Hilbert space has states that describe branes and Kaluza-Klein particles. The variables associated to imaginary roots play an important role in the reproduction of the mass of these branes and particles. Including branes corresponds, in this Hilbert space, to exciting a certain subset of the variables to higher Landau levels of an abstract particle in a magnetic field. Unfortunately, the masses of the branes are off by a factor of 2π, although their dependence on the metric is correct.

Many open questions remain:

1. What is the physical significance of the multiplicities of imaginary roots? The imaginary roots that we studied all have a multiplicity of $m = 8$, but, as we have seen in

section 4.3, all 8 generators that correspond to the same root also correspond to the same brane. Could this multiplicity be related to the multiplicity of the supersymmetric multiplets?

2. In section 5 we classified some interactions between branes and fluxes and between pairs of branes according to the inner product of the corresponding roots. We covered the cases $(\alpha|\beta) = 0, -1, -2$ for branes and fluxes, and the cases $(\alpha|\beta) = -1, -2, -4$ for pairs of branes. It would be interesting to study other values of the inner products.

3. We have argued in section 4.4 that a certain imaginary root is associated to a pair of Minkowski branes of different types. It would be interesting to relate the properties of the individual branes to the properties of the root.

4. It would be interesting to prove or disprove our general decomposition conjecture that every positive imaginary root can be decomposed as a sum of positive real roots. (See section 4.7.)

5. Can the process used in section 4.1 be generalized to other imaginary roots?

6. Can the process of passing one instanton through another, used in section 4.1, be generalized to triple commutators such as $[J^{+\alpha_1}, [J^{+\alpha_2}, J^{+\alpha_3}]]$?

7. In section 6 we explored various definitions for the subalgebra $U_\gamma \subset \hat{\mathbf{k}} \subset E_{10}$ associated with the root γ. It would be interesting to study the physical interpretation of U_γ and its relation to the supersymmetry that is preserved by the brane.

8. It would be interesting to extend the discussion to heterotic string theory where the Kac-Moody algebra DE_{18} plays a role [11]. It is intriguing that D7-branes can actually be created entirely with E_{10}, since after the lift from type-IIB to M-theory they correspond to imaginary roots, as in section 4. Therefore, it might be possible to rephrase the F-theory construction [71] entirely in terms of E_{10} variables. It would be interesting to find out how this works!

9. In the hamiltonian formulation discussed in section 7, can the zero-point energy be cancelled? Can the condition of total charge neutrality be incorporated?

10. Perhaps the most intriguing question is whether we can create an arbitrary collection of branes via a process as in section 4.4, or using the formalism of section 7. If true, it would mean that an arbitrary state of the universe can be described with the variables associated to imaginary roots of E_{10}!

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A. The singleton count of real and imaginary roots

In this appendix we will prove some of the theorems from section 3. We start with claim 3: a positive imaginary root has no negative n_i's. A positive real root has negative n_i's only if it is a permutation of $(1,-1,0,\dots,0)$.

Proof. If at least one $n_i = 0$, say for $i = 1$, then the root is in an E_9 subalgebra for which the roots are completely classified. They are the roots of E_8 plus an integer multiple of $(0,1,1,\dots,1)$. It is easy to verify that the theorem holds in that case. So we assume that all $n_i \neq 0$. Suppose without loss of generality that $n_k \leq n_{k+1} \leq \cdots \leq n_{10} < 0$ and $0 < n_1 \leq n_2 \leq \cdots \leq n_{k-1}$ for some $2 \leq k \leq 9$. Then

$$\begin{aligned}
\alpha^2 &= \frac{1}{9}\sum_{1\leq i<j\leq 10}(n_i-n_j)^2 - \frac{1}{9}\sum_{i=1}^{10} n_i^2 \\
&= \frac{1}{9}\sum_{1\leq i<j\leq k-1}(n_i-n_j)^2 + \frac{1}{9}\sum_{k\leq i<j\leq 10}(n_i-n_j)^2 + \frac{1}{9}\sum_{i\leq k-1}\sum_{j\geq k}(n_i+|n_j|)^2 - \frac{1}{9}\sum_{i=1}^{10} n_i^2 \\
&= \frac{1}{9}\sum_{1\leq i<j\leq k-1}(n_i-n_j)^2 + \frac{1}{9}\sum_{k\leq i<j\leq 10}(n_i-n_j)^2 + \frac{2}{9}\sum_{i\leq k-1}\sum_{j\geq k} n_i|n_j| + \\
&\quad + \frac{1}{9}\sum_{i=1}^{k-1}(10-k)n_i^2 + \frac{1}{9}\sum_{j=k}^{10}(k-2)n_i^2 \geq \frac{1}{9}[4(k-1)(11-k)-10] > 2\,.
\end{aligned}$$

$\square$

Next, we prove theorem 1: the only imaginary roots with a singleton count $s \geq 2$ are permutations of

$$\alpha = (0,1,1,1,1,1,1,1,1,1)\,, \qquad \alpha^2 = 0\,, \qquad s = 9$$

$$\begin{aligned}
\alpha &= (1,1,1,1,1,1,1,1,2,2)\,, \qquad \alpha^2 = 0\,, \qquad s = 8\\
\alpha &= (1,1,1,1,1,2,2,2,2,2)\,, \qquad \alpha^2 = 0\,, \qquad s = 5\\
\alpha &= (1,1,1,2,2,2,2,2,2,3)\,, \qquad \alpha^2 = 0\,, \qquad s = 3\\
\alpha &= (1,1,2,2,2,2,2,3,3,3)\,, \qquad \alpha^2 = 0\,, \qquad s = 2\\
\alpha &= (1,1,2,2,3,3,3,3,3,3)\,, \qquad \alpha^2 = 0\,, \qquad s = 2\\
\alpha &= (1,1,3,3,3,3,3,3,3,4)\,, \qquad \alpha^2 = 0\,, \qquad s = 2\\
\alpha &= (1,1,2,2,2,2,2,2,2,2)\,, \qquad \alpha^2 = -2\,, \qquad s = 2
\end{aligned}$$

and there is an infinite number of imaginary roots with singleton count $s = 1$.

Proof. According to the previous theorem, $n_i \geq 0$ for $i = 1, \ldots, 10$. If $n_i = 0$, for some n_i, then α is a root of an E_9 subalgebra. But the only imaginary roots of E_9 are given by

$$\alpha = (n,n,n,n,n,n,n,n,n)\,, \qquad \alpha^2 = 0\,.$$

This has a singleton count $s = 9$ for $n = 1$ and singleton count $s = 0$ for $n > 1$. So, suppose without loss of generality that $n_1 = n_2 = \cdots = n_s = 1$ and that $2 \leq n_{s+1} \leq n_2 \leq \cdots \leq n_{10}$ for $s \geq 1$. In order for α to be a root we need $s + \sum_{i=s+1}^{10} n_i \in 3\mathbb{Z}$. Then

$$\begin{aligned}
\alpha^2 &= \frac{1}{9} \sum_{1\leq i<j\leq 10} (n_i - n_j)^2 - \frac{1}{9}\sum_{i=1}^{10} n_i^2\\
&= \frac{1}{9} \sum_{s+1\leq i<j\leq 10} (n_i - n_j)^2 + \frac{s}{9}\sum_{i=s+1}^{10} (n_i - 1)^2 - \frac{1}{9}\sum_{i=s+1}^{10} n_i^2 - \frac{s}{9}\,.
\end{aligned}$$

If $s > 1$ we can write

$$\alpha^2 = \frac{1}{9} \sum_{s+1\leq i<j\leq 10} (n_i - n_j)^2 + \frac{s-1}{9}\sum_{i=s+1}^{10} \left(n_i - \frac{s}{s-1}\right)^2 - \frac{s}{s-1}\,.$$

There is only a finite number of sequences $2 \leq n_{s+1} \leq \cdots \leq n_{10}$ for which the righthand side is not positive. A quick exhaustive computer search yielded the 8 imaginary roots stated above. For $s = 1$ we get

$$\alpha^2 = \frac{1}{9} \sum_{2\leq i<j\leq 10} (n_i - n_j)^2 - \frac{2}{9}\sum_{i=2}^{10} n_i + \frac{8}{9} \tag{A.1}$$

and there is an infinite number of imaginary roots with $s = 1$ because for any given imaginary root α we can always change $n_i \to n_i + 1$ for all $i = 1, \ldots, 9$ and get a root with a smaller α^2. □

Finally, we prove theorem 2: the only real roots ($\alpha^2 = 2$) with a singleton count $s \geq 2$ are permutations of

$$\alpha = (0,0,0,0,0,0,0,1,1,1)\,, \qquad s = 3$$

$$
\begin{aligned}
\alpha &= (0,0,0,0,1,1,1,1,1,1)\,, & s&=6\\
\alpha &= (0,0,1,1,1,1,1,1,1,2)\,, & s&=7\\
\alpha &= (0,1,1,1,1,1,1,2,2,2)\,, & s&=6\\
\alpha &= (0,1,1,1,2,2,2,2,2,2)\,, & s&=3\\
\alpha &= (1,1,1,1,1,1,1,1,1,3)\,, & s&=9\\
\alpha &= (1,1,1,1,1,1,2,2,2,3)\,, & s&=6\\
\alpha &= (1,1,1,1,2,2,2,2,3,3)\,, & s&=4\\
\alpha &= (1,1,1,2,2,2,3,3,3,3)\,, & s&=3\\
\alpha &= (1,1,1,3,3,3,3,3,3,3)\,, & s&=3\\
\alpha &= (1,1,2,2,2,2,2,2,3,4)\,, & s&=2\\
\alpha &= (1,1,2,2,2,3,3,3,3,4)\,, & s&=2\\
\alpha &= (1,1,2,3,3,3,3,3,4,4)\,, & s&=2\\
\alpha &= (1,1,3,3,3,3,4,4,4,4)\,, & s&=2\\
\alpha &= (1,1,3,4,4,4,4,4,4,4)\,, & s&=2
\end{aligned}
$$

and there is an infinite number of imaginary roots with singleton count $s = 1$.

Proof. The proof is very similar to that of theorem 1. Note that to satisfy $\alpha^2 = 2$ in equation (A.1), we can start by fixing some difference, say $n_3 - n_2$, and pick an otherwise arbitrary sequence $2 \leq n_2 \leq \cdots \leq n_{10}$ such that $1 + \sum_2^{10} n_i$ is divisible by 3 and the righthand side of (A.1) is positive. (It is not hard to see that there are an infinite number of such sequences for any value of $n_3 - n_2$.) It is also easy to see that the righthand side is then an even integer (as it must, being an element of the E_{10} root lattice $\hat{Q}$). If we now change $n_i \to n_i + k$ for $i = 2, \ldots 10$ we see that we decrease the righthand side of (A.1) by $2k$. we can therefore find the appropriate k for which $\alpha^2 = 2$, and there is an infinite number of roots like that. □

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